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Integrability of type II superstrings on Ramond-Ramond backgrounds in various dimensions

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Abstract: We consider type II superstrings on AdS backgrounds with Ramond-Ramond flux in various dimensions. We realize the backgrounds as supercosets and analyze explicitly two classes of models: non-critical superstrings on AdS_{2d} and critical superstrings on $AdS_p \times S^p \times CY$. We work both in the Green-Schwarz and in the pure spinor formalisms. We construct a one-parameter family of flat currents (a Lax connection), leading to an infinite number of conserved non-local charges, which imply the classical integrability of both sigma-models. In the pure spinor formulation, we use the BRST symmetry to prove the quantum integrability of the sigma-model. We discuss how classical κ-symmetry implies one-loop conformal invariance. We consider the addition of space-filling D-branes to the pure spinor formalism.

Keywords: Superstrings and Heterotic Strings, Integrable Field Theories.

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1. Introduction and summary

Superstring theory on AdS backgrounds with Ramond-Ramond flux has not been quantized yet. The Green-Schwarz sigma-model on such backgrounds is an interacting two-dimensional conformal field theory. In the case of the type IIB superstring on $AdS_5 \times S^5$ [1], the authors of [2] showed that the sigma-model is invariant under a Yangian symmetry algebra and as a result is classically integrable. Their result relies on the realization of the background as a supercoset G/H, where G is a supergroup with a $\mathbb{Z}_4$ automorphism group and H is the $\mathbb{Z}_4$ fixed locus bosonic subgroup of G. Once uncovering this hidden symmetry, one can ask whether the Yangian algebra, derived for the $AdS_5 \times S^5$ background, is a general feature of superstrings on AdS backgrounds with RR flux.

We will address this question by looking at superstring theories on such backgrounds, both in the Green-Schwarz and the pure spinor formalisms. We will first construct sigma-model actions and find simple actions for the Green-Schwarz and the pure spinor superstrings, which hold in all dimensions. We will then show classical integrability of both sigma-models as well as quantum integrability of the pure spinor one.[1]

In general for the GS superstring, it is difficult to analyze the quantum sigma-model. This is because the quantization of the GS superstring is known only in the light-cone gauge and hence non-covariantly. Since the equations of motion of the GS superstring do not provide a propagator for the θ's, the calculations in worldsheet perturbation theory are problematic. On the other hand, the pure spinor sigma-model can be quantized in a straightforward manner, since it contains additional terms that break explicitly the GS κ-symmetry and introduce propagators for all the variables. Hence, we will be able to show that our models are gauge invariant and BRST invariant at all orders in the worldsheet perturbation theory using the methods of [10].

We will consider explicitly two classes of models: Type II non-critical superstrings on AdS_{2d}, for $d = 1, 2, 3$, and Type II critical superstrings on $AdS_p \times S^p \times CY_{5-p}$, for $p = 2, 3$.

The first class of models are strongly coupled two-dimensional CFTs. The sigma-model coupling, given by the curvature of AdS, is fixed to a finite value of order one in string units, and the theory cannot be analyzed perturbatively. The worldsheet variables for the non-critical superstrings and in particular their pure spinor spaces have been derived in [11] by mapping the RNS formulation of the linear dilaton background to the covariant one.[2]

[1]Recently, these kinds of supercoset sigma-models have received attention regarding their integrability properties, see for example [3 – 9].

[2]See also [12] for the hybrid formulation of the linear dilaton background and [13 – 15] about lower-dimensional pure spinor superstrings.

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In the second class of models, the sigma-model will describe the non-compact part $AdS_p \times S^p$ of critical superstrings on ten-dimensional backgrounds. Unlike the previous non-critical string case, the curvature of AdS is a modulus. Thus, one can take the limit in which the curvature is small and the sigma-model is weakly coupled and can be studied perturbatively.

All our models are realized as nonlinear sigma-models on supercosets G/H, where the supergroup G has a $\mathbb{Z}_4$ automorphism, whose invariant locus is H. A crucial property of sigma-models on such supercosets is their classical integrability. In order to exhibit the integrability of the sigma-models, we have to construct an infinite number of conserved charges [7, 5]. Furthermore, for the charges to be physical they have to be κ-invariant and BRST invariant in the Green-Schwarz and in the pure spinor formalisms, respectively. The first step in the construction of the charges is to find a one-parameter family of currents $a(\mu)$ satisfying the flatness condition

$$da(\mu) + a(\mu) \wedge a(\mu) = 0 \ . \tag{1.1}$$

One then constructs the Wilson line

$$U_{(\mu)}(x,t;y,t) = \mathrm{P}\exp\left(-\int_{(y,t)}^{(x,t)} a(\mu)\right) , \tag{1.2}$$

and obtains the infinite set of non-local charges Q_n by expanding

$$U_{(\mu)}(\infty,t;-\infty,t) = 1 + \sum_{n=1}^{\infty} \mu^n Q_n \ . \tag{1.3}$$

The conservation of Q_n is implied by the flatness of $a(\mu)$ (provided $a(\mu)$ vanishes at $\pm\infty$).[3] This is valid for a sigma model on a plane. In the closed string case, we need to impose periodic boundary conditions and hence consider a slightly different invariant — the trace of the Wilson loop.

The first two charges Q_1 and Q_2 generate the Yangian algebra, which is a symmetry algebra underlying the type II superstrings propagating on the AdS backgrounds with Ramond-Ramond fluxes in various dimensions. Moreover, in the pure spinor formalism one can see that this symmetry holds also at the quantum sigma-model level. This has been shown by Berkovits in the $AdS_5 \times S^5$ background in [10]. We will show that quantum integrability of the pure spinor action holds also in the lower-dimensional cases. In the case of type IIB superstrings propagating on $AdS_5 \times S^5$, a similar Yangian algebra has been identified in the free field theory limit of $\mathcal{N} = 4$ SYM at large N_c [17]. We expect that a similar structure underlies the field theory duals in various dimensions.

Note, that the Yangian algebra suggests the existence of an affine Kac-Moody algebra [18]. This is to be contrasted with NS-NS backgrounds, where the affine algebra comes in two copies, one left- and one right-moving, while in the case of RR backgrounds there

[3]Establishing the existence of the Lax connection (1.1) is the first step towards the solution of the nonlinear sigma-model. In particular, this technique has been fully exploited in [16] to find the classical spectrum of the GS type IIB superstring on $AdS_5 \times S^5$.

would be only a single copy of such an algebra. The question arises whether this symmetry is sufficient for solving for the spectrum of the superstring.

The paper is organized as follows. Throughout most of the paper, we analyze in general the structure of superstrings sigma-models on supercosets with $\mathbb{Z}_4$ automorphisms. As we will see, most of their properties are algebraic and do not rely on the particular choice of the supercoset. In section 2 we introduce the classical κ-invariant Green-Schwarz sigma-model and find a one-parameter family of flat currents (Lax connection). This leads to an infinite number of conserved non-local charges and shows classical integrability of the sigma-model. In section 3 we introduce the pure spinor action and compute the one-parameter family of flat currents, which is different from the GS one. We describe also the various pure spinor spaces that we use in the various dimensions. At the end of the section, we discuss the addition of open string boundary conditions to the pure spinor sigma-model. In section 4, we study the pure spinor sigma-model at the quantum level and show that it is gauge invariant and BRST invariant at all orders in perturbation theory and argue that for AdS_2 these properties hold non-perturbatively as well. By using BRST symmetry, we then prove quantum integrability. In section 5, we study one-loop conformal invariance of the GS sigma-model and then describe the various specific backgrounds and their supercoset realizations in section 6. In appendices A and B we collected some technical details of the GS and pure spinor computations, while in appendix C we describe the various supergroups and their notations. In appendix D we review the supergravity solution of non-critical $AdS_5 \times S^1$ of [19] and find a curious result about the higher curvature corrections to this solution.

2. Integrability of Green-Schwarz superstrings on RR backgrounds

In this section we will consider the integrability properties of Green-Schwarz superstrings on the background of a supercoset G/H with only RR-flux, where G is a supergroup with a $\mathbb{Z}_4$ automorphism whose invariant locus is the subgroup H. We will construct the Green-Schwarz action and derive the family of flat connections leading to an infinite number of conserved non-local charges.[4] The κ-invariance of the currents in the GS formalism will follow from the BRST invariance of the non-local currents in the pure spinor formalism that we will prove in the next section as explained in [21]. The notations about supergroups are summarized in the appendix. The discussion in this section will be at a formal level, while we will specialize to the particular backgrounds in sections 5 and 6.

2.1 The Green-Schwarz sigma-model

We will be interested in sigma-models whose target space is the coset G/H, where G is a supergroup with a $\mathbb{Z}_4$ automorphism and the subgroup H is the invariant locus of this automorphism. The super Lie algebra $\mathcal{G}$ of G can be decomposed into the $\mathbb{Z}_4$ automorphism invariant spaces $\mathcal{G} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where the subscript keeps track of the $\mathbb{Z}_4$ charge

[4] Our construction will be covariant. In the case of $AdS_5 \times S^5$, it has been shown in [20] that the Green-Schwarz sigma-model is still integrable after gauge fixing of κ-symmetry and reparametrization invariance.

and in particular $\mathcal{H}_0$ is the algebra of the subgroup H. This decomposition satisfies the algebra ($i = 1, \ldots, 3$)

$$[\mathcal{H}_0, \mathcal{H}_0] \subset \mathcal{H}_0\,, \quad [\mathcal{H}_0, \mathcal{H}_i] \subset \mathcal{H}_i\,, \quad [\mathcal{H}_i, \mathcal{H}_j] \subset \mathcal{H}_{i+j \text{ mod } 4}. \tag{2.1}$$

and the only non-vanishing supertraces[5] are

$$\langle \mathcal{H}_i \mathcal{H}_j \rangle \neq 0\,,\ i + j = 0 \text{ mod } 4 \quad (i, j = 0, \ldots, 3)\ . \tag{2.2}$$

We will denote the bosonic generators in $\mathcal{G}$ by $T_{[ab]} \in \mathcal{H}_0$, $T_a \in \mathcal{H}_2$, and the fermionic ones by $T_\alpha \in \mathcal{H}_1$, $T_{\hat{\alpha}} \in \mathcal{H}_3$.

The worldsheet fields are the maps $g : \Sigma \to G$ and dividing by the subgroup H is done by gauging the subgroup H acting from the right by $g \simeq gh$, $h \in H$. The sigma-model is further constrained by the requirement that it be invariant under the global symmetry $g \to \hat{g} g$, $\hat{g} \in G$. The left-invariant current is defined as

$$J = g^{-1} dg\,, \tag{2.3}$$

which satisfies the Maurer-Cartan equation

$$dJ + J \wedge J = 0\ . \tag{2.4}$$

This current can be decomposed according to the $\mathbb{Z}_4$ grading of the algebra $J = J_0 + J_1 + J_2 + J_3$ and the Maurer-Cartan equation splits into

$$dJ_0 + J_0 \wedge J_0 + J_1 \wedge J_3 + J_2 \wedge J_2 + J_3 \wedge J_1 = 0\,, \tag{2.5}$$
$$dJ_1 + J_0 \wedge J_1 + J_1 \wedge J_0 + J_2 \wedge J_3 + J_3 \wedge J_2 = 0\,, \tag{2.6}$$
$$dJ_2 + J_0 \wedge J_2 + J_1 \wedge J_1 + J_2 \wedge J_0 + J_3 \wedge J_3 = 0\,, \tag{2.7}$$
$$dJ_3 + J_0 \wedge J_3 + J_1 \wedge J_2 + J_2 \wedge J_1 + J_3 \wedge J_0 = 0\ . \tag{2.8}$$

These currents are manifestly invariant under the global symmetry, which acts by left multiplication. Under the gauge transformation, which acts by right multiplication, they transform as

$$\delta J = d\Lambda + [J, \Lambda]\,, \quad \Lambda \in \mathcal{H}_0\ . \tag{2.9}$$

Using the above properties of the algebra $\mathcal{G}$ and the requirement of gauge invariance leads to the GS action (in the following we will use J_i both to denote the 1-form currents in the target space as well as their pullback to the worldsheet)

$$S_{\text{GS}} = \frac{1}{4} \int \langle J_2 \wedge *J_2 + J_1 \wedge J_3 \rangle = \frac{1}{4} \int d^2\sigma \langle \sqrt{h} h^{mn} J_{2m} J_{2n} + \epsilon^{mn} J_{1m} J_{3n} \rangle\,, \tag{2.10}$$

where $m, n = 1, 2$ are worldsheet indices. A $J_0 \wedge *J_0$ term does not appear because of gauge invariance, while the term $J_1 \wedge *J_3$ breaks κ-symmetry and therefore cannot be included

[5]The supertrace of a supermatrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is defined as $\langle M \rangle = \text{Str} M = \text{tr} A - (-1)^{\text{deg} M} \text{tr} D$, where $\text{deg} M$ is 0 for Grassmann even matrices and 1 for Grassmann odd ones.

in the GS action. The first and second terms in the action are the kinetic and Wess-Zumino terms, respectively. The coefficient of the Wess-Zumino term is determined using κ-symmetry as shown in the next paragraph. For a particular choice of the supergroup, this GS action reproduces the GS action on AdS_2 background constructed in [22] and the GS action on $AdS_5 \times S^5$ [1].

Let us verify now that the action is indeed invariant under κ-symmetry. It is convenient to parameterize the κ-transformation by [1]

$$\delta_\kappa x_i \equiv \delta_\kappa X^M J_{iM} \quad , \tag{2.11}$$

where the index M runs over the target superspace indices and X^M are the superspace coordinates, while $i = 1, \ldots, 3$ denotes the $\mathbb{Z}_4$ grading. Since $J_i = dX^M J_{iM}$ we obtain the following transformations of the currents

$$\delta_\kappa J_2 = d\delta_\kappa x_2 + [J_0, \delta_\kappa x_2] + [J_2, \delta_\kappa x_0] + [J_1, \delta_\kappa x_1] + [J_3, \delta_\kappa x_3]\,, \tag{2.12}$$
$$\delta_\kappa J_1 = d\delta_\kappa x_1 + [J_0, \delta_\kappa x_1] + [J_1, \delta_\kappa x_0] + [J_2, \delta_\kappa x_3] + [J_3, \delta_\kappa x_2]\,, \tag{2.13}$$
$$\delta_\kappa J_0 = d\delta_\kappa x_0 + [J_0, \delta_\kappa x_0] + [J_1, \delta_\kappa x_3] + [J_2, \delta_\kappa x_2] + [J_3, \delta_\kappa x_1]\,, \tag{2.14}$$
$$\delta_\kappa J_3 = d\delta_\kappa x_3 + [J_0, \delta_\kappa x_3] + [J_1, \delta_\kappa x_2] + [J_2, \delta_\kappa x_1] + [J_3, \delta_\kappa x_0]\;. \tag{2.15}$$

Using these transformations and taking into account the Maurer-Cartan equations, the κ-transformation of the actions is

$$\begin{aligned}\delta_\kappa S_{\rm GS} = \frac{1}{4}\int d^2\sigma \langle & \epsilon^{mn}\partial_m (J_{3n}\delta_\kappa x_1 - J_{1n}\delta_\kappa x_3) + \delta_\kappa(\sqrt{h}h^{mn}) J_{2m} J_{2n} + \\ & + 2\sqrt{h}h^{mn}(J_{2m}\partial_n \delta_\kappa x_2 + [J_{2m}, J_{0n}]\delta_\kappa x_2) + \epsilon^{mn}([J_{1m}, J_{1n}] - [J_{3m}, J_{3n}])\delta_\kappa x_2 - \\ & - 2(\sqrt{h}h^{mn} + \epsilon^{mn})[J_{1n}, J_{2m}]\delta_\kappa x_1 + 2(\sqrt{h}h^{mn} - \epsilon^{mn})[J_{2m}, J_{3n}]\delta_\kappa x_3\rangle\;. \end{aligned} \tag{2.16}$$

The κ-transformation is parameterized by

$$\delta_\kappa x_2 = 0\,, \quad \delta_\kappa x_1 = [J_{2m}, \kappa_3^m]\,, \quad \delta_\kappa x_3 = [J_{2m}, \kappa_1^m]\,, \tag{2.17}$$

where $\kappa_3^m \in \mathcal{H}_3$ and $\kappa_1^m \in \mathcal{H}_1$. By substituting this and expressing the result in terms of the structure constants and the Cartan metric η one finally has

$$\begin{aligned}\delta_\kappa S_{\rm GS} = \frac{1}{4}\int d^2\sigma \Big[& \epsilon^{mn}\langle \partial_m (J_{3n}\delta_\kappa x_1 - J_{1n}\delta_\kappa x_3)\rangle + \delta_\kappa(\sqrt{h}h^{mn})\eta_{ab} J^a_{2m} J^b_{2n} + \\ & + 4\sqrt{h}(P_+^{mn}\eta_{\hat\beta\beta} f^{\hat\beta}_{\alpha a} f^{\beta}_{b\hat\alpha} J^{\alpha}_{1n}\kappa_3^{p\hat\alpha} - P_-^{mn}\eta_{\beta\hat\beta} f^{\beta}_{a\hat\alpha} f^{\hat\beta}_{b\alpha} J^{\hat\alpha}_{3n}\kappa_1^{p\alpha}) J^a_{2m} J^b_{2p}\Big]\,, \end{aligned} \tag{2.18}$$

where we have defined the projectors $P_\pm^{mn} = \frac{1}{2}(h^{mn} \pm \frac{1}{\sqrt{h}}\epsilon^{mn})$. Since $\delta_\kappa(\sqrt{h}h^{mn})$ should be symmetric and traceless and not Lie-algebra valued, we have to require that

$$\eta_{\beta\hat\beta}\left(f^{\beta}_{a\hat\alpha} f^{\hat\beta}_{b\alpha} + f^{\beta}_{b\hat\alpha} f^{\hat\beta}_{a\alpha}\right) = c_{\alpha\hat\alpha}\eta_{ab} \tag{2.19}$$

for some matrix $c_{\alpha\hat\alpha}$. Then one obtains

$$\delta_\kappa(\sqrt{h}h^{mn}) = 4\sqrt{h} c_{\alpha\hat\alpha}(P_-^{mp} J^{\hat\alpha}_{3p}\kappa_1^{n\alpha} - P_+^{mp} J^{\alpha}_{1p} k_3^{n\hat\alpha})\,, \tag{2.20}$$

which is automatically symmetric in a and b if we require that

$$\kappa_1^m = P_-^{mn}\kappa_{1n}\,, \quad \kappa_3^m = P_+^{mn}\kappa_{3n} \tag{2.21}$$

since $P_\pm^{mp}P_\pm^{nq} = P_\pm^{np}P_\pm^{mq}$. It is also traceless because $P_-^{nm}\kappa_{1n} = P_+^{nm}\kappa_{3n} = 0$.

The relation (2.19), required for κ-symmetry, is a condition on the structure constants of the supergroup. This condition is equivalent to the torsion constraints of type II supergravity in various dimensions.[6] In ten dimensions, by requiring κ-symmetry of the GS action one finds the constraints of ten-dimensional supergravity. In the non-critical superstring, we get for backgrounds of this type one of the supergravity constraints. In appendix A, we work out the relation between (2.19) and the torsion constraints.

2.2 Classical integrability of the Green-Schwarz sigma-model

In the following we construct a one-parameter family of flat currents (2.27), (2.29) that imply the existence of an infinite number of conserved non-local charges, thus showing that the GS sigma-model is classically integrable. The κ-invariance of these currents will not be checked, but it should follow from the BRST invariance of the corresponding pure-spinor currents shown in appendix B.2.

The equation of motion and constraints for the currents that follow from the action (2.10) read

$$\begin{aligned} d{*}J_2 &= -J_0\wedge *J_2 - *J_2\wedge J_0 + J_1\wedge J_1 - J_3\wedge J_3\,, && (2.22)\\ 0 &= J_1\wedge J_2 + J_2\wedge J_1 + *J_1\wedge J_2 + J_2\wedge *J_1, && (2.23)\\ 0 &= J_2\wedge J_3 + J_3\wedge J_2 - J_2\wedge *J_3 - *J_3\wedge J_2,\ . && (2.24)\end{aligned}$$

We are looking for a one parameter family of flat connections $D = d + a(\mu)$, satisfying the zero curvature condition $D^2 = 0$ or in other words

$$da(\mu) + a(\mu)\wedge a(\mu) = 0, \tag{2.25}$$

where the right-invariant current $a(\mu)$ is usually referred to as the Lax connection and μ as the spectral parameter. In order to facilitate the comparison with the pure spinor flat current, we will switch to the left-invariant current $A = g^{-1}ag$ which satisfies the equation

$$dA + A\wedge A + J\wedge A + A\wedge J = 0\ . \tag{2.26}$$

Following [2] we will consider a current composed of the currents for which the exterior derivative is known:

$$A = \alpha J_2 + \beta{*}J_2 + \gamma J_1 + \delta J_3\ . \tag{2.27}$$

[6]By “type II” supergravity in dimension D we mean a theory with as many gravitini as the ones we would get by compactifying ten-dimensional type II supergravity on a Calabi-Yau of real dimension $10-D$.

Substituting this in (2.26) and using the equation of motion, the constraints and the Maurer-Cartan equations yields the equations[7]

$$\begin{aligned} \beta - \alpha + \gamma^2 + 2\gamma &= 0\,, & -\alpha - \beta + \delta^2 + 2\delta &= 0\,, \\ -\gamma + (\alpha - \beta)\delta + \alpha - \beta + \delta &= 0\,, & -\delta + (\alpha + \beta)\gamma + \alpha + \beta + \gamma &= 0\,, \\ \alpha^2 - \beta^2 + 2\alpha &= 0\,, & \gamma\delta + \gamma + \delta &= 0\,, \end{aligned} \tag{2.28}$$

whose two one-parameter families of solutions are

$$\begin{aligned} \alpha &= 2\sinh^2\mu\,, \quad \beta = 2\sinh\mu\cosh\mu\,, \quad \gamma = -(1+e^{-\mu})\,, \quad \delta = -(1+e^{\mu})\,, \\ \alpha &= 2\sinh^2\mu\,, \quad \beta = -2\sinh\mu\cosh\mu\,, \quad \gamma = e^{\mu} - 1\,, \quad \delta = e^{-\mu} - 1\,, \end{aligned} \tag{2.29}$$

where $-\infty < \mu < \infty$.

For the second family, an infinite set of conserved charges can be obtained using the expansion of the solution about $\mu = 0$

$$a = \mu(j_1 - j_3 - 2{*}j_2) + \mu^2\left(2j_2 + \frac{1}{2}j_1 + \frac{1}{2}j_3\right) + O(\mu^3)\,, \tag{2.30}$$

where the j_i denote the right-invariant currents gJ_ig^{-1}. We can then introduce the monodromy matrix, which is the Wilson line of the flat connection

$$U_C = \mathrm{P}\exp\left(-\int_C a\right) = 1 + \sum_{n=1}^{\infty}\mu^n Q_n\,, \tag{2.31}$$

whose expansion around $\mu = 0$ leads to the conserved charges Q_n. The first two conserved charges are[8]

$$Q_1 = -\int_C (j_1 - j_3 - 2{*}j_2)\,, \tag{2.32}$$

$$Q_2 = -\int_C\left(2j_2 + \frac{1}{2}j_1 + \frac{1}{2}j_3\right) + \int_C [j_1(x) - j_3(x) - 2{*}j_2(x)]\int_o^x (j_1 - j_3 - 2{*}j_2). \tag{2.33}$$

The former is local and is expected to be one of the Noether currents of the sigma-model. The latter is non-local. The other charges can be generated by repetitive Poisson brackets of Q_2 and together they form a classical Yangian. The Lax connection is the starting point for the solution of the classical sigma-model (see e.g. [16]).

We will not argue that the integrability property is preserved at the quantum level. This will be shown in the pure spinor formalism.

[7]Our currents are related to the currents in [2] by $p = -j_2$, $q = -(j_1 + j_3)$ and $q' = j_1 - j_3$ so these equations are related to the ones in [2] by $\alpha = -\widetilde{\alpha}$, $\beta = -\widetilde{\beta}$, $\gamma = \widetilde{\delta} - \widetilde{\gamma}$ and $\delta = -(\widetilde{\gamma} + \widetilde{\delta})$, where the tilded variables refer to the same untilded variables in [2].

[8]In the notation of [2] the integrand of Q_1 is proportial the Noether current $p + \frac{1}{2}{*}q'$.

3. Integrability of pure spinor superstrings on RR backgrounds

In this section we will consider pure spinor superstrings on coset super-manifolds G/H, where the supergroup G possesses a $\mathbb{Z}_4$ automorphism whose invariant locus is the subgroup H. The cosets we will consider will be limited to backgrounds which have only RR-flux. We will first discuss the various pure spinor spaces in the different spacetime dimensions, then construct the BRST invariant pure spinor action and the infinite set of BRST invariant non-local charges, hence exhibiting the classical integrability of the pure spinor superstrings. In the following section we will prove that these pure spinor superstrings are also integrable at the quantum level. Towards the end of the section we will discuss the inclusion of D-branes in the pure spinor superstrings.

The pure spinor formalism for the ten-dimensional superstring [23] has been well established. In lower dimensions, there have been different interpretations of the pure spinor superstring action. In some cases it has been argued that it describes the non-critical superstring [11], in other cases it has been argued to describe the non-compact sector of a ten-dimensional superstring compactified on a CY manifold [24, 14, 13]. In this section, we will focus on the algebraic properties of the pure spinor formulation of the superstring of a supercoset sigma-model.

3.1 Pure spinor spaces in two, four and six dimensions

In this subsection we will present the definition of the pure spinor spaces in lower-dimensional superstrings. The definition of the pure spinors that Cartan and Chevalley give in even dimension $d = 2n$ is that $\lambda\sigma^{m_1 \ldots m_j}\lambda = 0$ for $j < n$, so that the pure spinor bilinear reads [25, 26][9]

$$\lambda^\alpha \lambda^\beta = \frac{1}{n!2^n}\sigma^{\alpha\beta}_{m_1 \ldots m_n}(\lambda\sigma^{m_1 \ldots m_n}\lambda), \tag{3.1}$$

where $\sigma^{m_1 \ldots m_j}$ is the antisymmetrized product of j Pauli matrices. This definition of the pure spinor space in $d = 2, 4, 6$ dimensions is trivially realized by an SO(d) Weyl spinor.

In all our cases the lower-dimensional pure spinors will be Weyl spinors. In some cases we will need more than just one pure spinor to construct a consistent string theory. In particular, our pure spinor spaces are dictated by the realization of the supersymmetry algebra for the type II superstring.[10] Indeed, we will use the same pure spinor spaces in $2p$ dimensions to describe the ghost sector of both the non-compact sector of the type II superstring on $AdS_p \times S^p \times CY_{5-p}$ and of the $2p$ dimensional non-critical type II superstring. These latter models have been introduced in [11], where a field redefinition has been constructed that maps the RNS formulation to the pure spinor formulation of the non-critical superstring in the linear dilaton backgrounds. The crucial feature of these lower-dimensional pure spinor spaces is that, like in the ten-dimensional case, the product of two pure spinors is still proportional to the middle dimensional form, according to 3.1. Let us discuss the various dimensions in detail.

[9] See also [27].

[10] These lower dimensional pure spinors spaces have been introduced in [24, 14, 28], in the context of the Calabi-Yau compactification of the ten-dimensional pure spinor superstring.

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Two-dimensional superstring

The left moving sector of Type II superstrings in two dimensions realizes $\mathcal{N} = (2,0)$ spacetime supersymmetry with 2 real supercharges Q_α, both of which are spacetime MW spinors of the same chirality, which are related by an SO(2) R-symmetry transformation (α is not a spinor index in this case, but just enumerates supercharges of the same chirality). The corresponding superderivatives are denoted by D_α. The supersymmetry algebra reads

$$\{D_\alpha, D_\beta\} = -\delta_{\alpha\beta} P^+,$$

where $P^\pm$ are the holomorphic (antiholomorphic) spacetime direction of AdS_2. The pure spinors are defined such that $\lambda^\alpha D_\alpha$ is nilpotent, so that the pure spinor condition in two dimensions reads

$$\lambda^\alpha \lambda^\beta \delta_{\alpha\beta} = 0, \tag{3.2}$$

which is solved by one Weyl spinor. The pure spinor bilinear reads

$$\lambda^\alpha \lambda^\beta = \frac{1}{2}(\tau_a)^{\alpha\beta}(\lambda^\gamma \tau^a_{\gamma\delta}\lambda^\delta), \tag{3.3}$$

where the index a takes the values $1,3$. In two dimensions the off-diagonal blocks of the gamma matrices are one dimensional matrices, so the relation (3.1) still holds.[11]

Four-dimensional superstring

In four dimensions, the left moving sector of the type II superstring realizes $\mathcal{N} = 1$ supersymmetry, which in terms of the superderivatives D_A in the Dirac form reads

$$\{D_A, D_B\} = -2(C\Gamma^m)P_m, \tag{3.4}$$

where C is the charge conjugation matrix and $A = 1, \ldots, 4$. Requiring nilpotence of $\lambda^A D_A$ specifies the four-dimensional pure spinor constraint

$$\lambda^A (C\Gamma^m)_{AB} \lambda^B = 0. \tag{3.5}$$

If we expand the pure spinor bilinear in terms of the four dimensional gamma matrices we find then $\lambda^A\lambda^B = \frac{1}{4}(C\Gamma_{mn})^{AB}(\lambda C\Gamma^{mn}\lambda)$. Sometimes it will be convenient to use the Weyl notation for the spinors, under which the pure spinor is represented by a pair of Weyl and anti-Weyl spinors $(\lambda^\alpha, \lambda^{\dot\alpha})$, subject to the constraint

$$\lambda^\alpha \lambda^{\dot\alpha} = 0. \tag{3.6}$$

The pure spinor bilinear then reads

$$\lambda^\alpha\lambda^\beta = \frac{1}{8}\sigma^{\alpha\beta}_{mn}(\lambda\sigma^{mn}\lambda), \qquad \lambda^{\dot\alpha}\lambda^{\dot\beta} = \frac{1}{8}\sigma^{\dot\alpha\dot\beta}_{mn}(\lambda\sigma^{mn}\lambda), \tag{3.7}$$

[11] The notations here are slightly different from the ones in [11]. In particular, if we denote by $\widetilde{\lambda}^i$ the pure spinor in that paper, we have $\widetilde{\lambda}^1 = \frac{1}{\sqrt{2}}(\lambda^1 + i\lambda^2)$ and $\widetilde{\lambda}^2 = \frac{1}{\sqrt{2}}(\lambda^1 - i\lambda^2)$. Anyway, the pure spinor space is identical to the one considered there.

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Six-dimensional superstring

In six dimensions, the left moving sector of the type II superstring realizes $\mathcal{N} = (1,0)$ supersymmetry, with eight real supercharges. Naively, one would expect that one supercharge Q_α in the $\bar{\mathbf{4}}$ of SO(6) could do the job. However, due to CPT invariance and the pseudo-reality of the Weyl irrep, it is impossible to realizes the supersymmetry algebra with just one copy of supercharges[12] and we have to introduce two supercharges Q^i_α in the $\bar{\mathbf{4}}$ of SO(6), which form a doublet of an auxiliary SU(2) outer automorphism. In terms of the superderivatives D^i_α the supersymmetry algebra reads

$$\{D^i_\alpha, D^j_\beta\} = \epsilon^{ij}\sigma^m_{\alpha\beta}P_m, \tag{3.8}$$

where ϵ^{ij} is the invariant tensor of SU(2). It is clear now that the six-dimensional pure spinor consists of a Weyl spinor λ^α_i which is also a doublet with respect to the auxiliary SU(2). If we demand the nilpotence of $\lambda^\alpha_i D^i_\alpha$ we then find the pure spinor constraint

$$\epsilon^{ij}\lambda^\alpha_i\sigma^m_{\alpha\beta}\lambda^\beta_j = 0. \tag{3.9}$$

If we expand the symmetric bispinor constructed out of a pure spinor bilinear, using representation theory we find once again that only the middle-dimensional form is present

$$\lambda^\alpha_i\lambda^\beta_j = \frac{1}{3!16}\sigma^{\alpha\beta}_{mnp}\sigma^{ab}_{ij}(\lambda\sigma^{mnp}\sigma_{ab}\lambda), \tag{3.10}$$

where σ^{ab}_{ij} is the two by two SU(2) generator in the fundamental representation, given by the antisymmetrized product of two SU(2) Pauli matrices.

3.2 The pure spinor sigma-model

The worldsheet action in the pure spinor formulation of the superstring consists of a matter and a ghost sector. The worldsheet metric is in the conformal gauge and there are no reparameterization ghosts. The matter fields are written in terms of the left-invariant currents $J = g^{-1}\partial g$, $\bar{J} = g^{-1}\bar{\partial} g$, where $g : \Sigma \to G$, and decomposed according to the invariant spaces of the $\mathbb{Z}_4$ automorphism:

$$J = J_0 + J_1 + J_2 + J_3 \tag{3.11}$$

and similarly for the anti-holomorphic component $\bar{J}$, where the notations are the same as in section 2. The Lie algebra-valued pure spinor fields and their conjugate momenta are defined as in [10]

$$\lambda = \lambda^\alpha T_\alpha\,, \quad w = w_\alpha\eta^{\alpha\hat{\alpha}}T_{\hat{\alpha}}\,, \quad \bar{\lambda} = \bar{\lambda}^{\hat{\alpha}}T_{\hat{\alpha}}\,, \quad \bar{w} = \bar{w}_{\hat{\alpha}}\eta^{\alpha\hat{\alpha}}T_\alpha\,, \tag{3.12}$$

where we decomposed the fermionic generators T of the super Lie algebra $\mathcal{G}$ according to their $\mathbb{Z}_4$ gradings $T_\alpha \in \mathcal{H}_1$ and $T_{\hat{\alpha}} \in \mathcal{H}_3$ and used the inverse of the Cartan metric $\eta^{\alpha\hat{\alpha}}$. The

[12] A simple manifestation of this fact is the following. The six-dimensional Pauli matrices $\sigma^m_{\alpha\beta}$ are four by four antisymmetric matrices. Therefore the naive supersymmetry algebra $\{Q_\alpha, Q_\beta\} = \sigma^m_{\alpha\beta}P_m$ does not make sense in six dimensions.

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spinor indices here are just a reminder, the unhatted ones refer to left moving quantities, the hatted ones to right moving ones. The choice of spinor representations depends on the particular supercoset in discussion and will be explained in section 6 for each specific model. Using these conventions, the pure spinor currents are defined by

$$N = -\{w, \lambda\}\,, \quad \bar{N} = -\{\bar{w}, \bar{\lambda}\}\,, \tag{3.13}$$

which generate in the pure spinor variables the Lorentz transformations that correspond to left-multiplication by elements of H. $N, \bar{N} \in \mathcal{H}_0$ so they indeed act on the tangent-space indices α and $\hat{\alpha}$ of the pure spinor variables as the Lorentz transformation. The pure spinor constraint reads

$$\{\lambda, \lambda\} = 0, \qquad \{\bar{\lambda}, \bar{\lambda}\} = 0\ . \tag{3.14}$$

The sigma-model should be invariant under the global transformation $\delta g = \Sigma g$, $\Sigma \in \mathcal{G}$. J and $\bar{J}$ are invariant under this global symmetry. The sigma-model should also be invariant under the gauge transformation

$$\begin{aligned} \delta_\Lambda J &= \partial\Lambda + [J, \Lambda]\,, \quad \delta_\Lambda \bar{J} = \bar{\partial}\Lambda + [\bar{J}, \Lambda] \quad \delta_\Lambda \lambda = [\lambda, \Lambda]\,, \quad \delta_\Lambda w = [w, \Lambda]\,, \\ \delta_\Lambda \bar{\lambda} &= [\bar{\lambda}, \Lambda]\,, \qquad\quad \delta_\Lambda \bar{w} = [\bar{w}, \Lambda]\,, \end{aligned} \tag{3.15}$$

where $\Lambda \in \mathcal{H}_0$. The most general sigma-model with these properties is

$$S = \int d^2z \langle \alpha J_2 \bar{J}_2 + \beta J_1 \bar{J}_3 + \gamma J_3 \bar{J}_1 + w\bar{\partial}\lambda + \bar{w}\partial\bar{\lambda} + N\bar{J}_0 + \bar{N}J_0 + aN\bar{N}\rangle\,, \tag{3.16}$$

where α, β, γ, a are numerical coefficients that we will shortly determine.

The accompanying BRST operator is (see appendix B.1)

$$Q_B = \oint \langle dz\lambda J_3 + d\bar{z}\bar{\lambda}\bar{J}_1\rangle\,, \tag{3.17}$$

which generates the following BRST transformations

$$\begin{aligned} \delta_B J_j &= \delta_{j+3,0}\partial(\epsilon\lambda) + [J_{j+3}, \epsilon\lambda] + \delta_{j+1,0}\partial(\epsilon\bar{\lambda}) + [J_{j+1}, \epsilon\bar{\lambda}], \\ \delta_B \bar{J}_j &= \delta_{j+3,0}\bar{\partial}(\epsilon\lambda) + [\bar{J}_{j+3}, \epsilon\lambda] + \delta_{j+1,0}\bar{\partial}(\epsilon\bar{\lambda}) + [\bar{J}_{j+1}, \epsilon\bar{\lambda}], \\ \delta_B w &= -J_3\epsilon, \qquad \delta_B \bar{w} = -\bar{J}_1\epsilon, \\ \delta_B N &= [J_3, \epsilon\lambda], \quad \delta_B \bar{N} = [\bar{J}_1, \epsilon\bar{\lambda}]. \end{aligned} \tag{3.18}$$

The coefficients of the various terms in the action are determined by requiring that the action be BRST invariant (the details can be found in appendix B.1). The BRST-invariant sigma-model thus obtained is

$$S = \int d^2z \left\langle \frac{1}{2}J_2\bar{J}_2 + \frac{1}{4}J_1\bar{J}_3 + \frac{3}{4}J_3\bar{J}_1 + w\bar{\partial}\lambda + \bar{w}\partial\bar{\lambda} + N\bar{J}_0 + \bar{N}J_0 - N\bar{N}\right\rangle \tag{3.19}$$

for all dimensions and this of course matches the critical $AdS_5 \times S^5$ considered in [10] as well.

Let us briefly comment on the relation between the pure spinor action (3.19) and the GS action (2.10). The latter, when written in conformal gauge, reads

$$S_{\rm GS} = \int d^2z \langle \frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 - \frac{1}{4} J_3 \bar{J}_1 \rangle \ . \tag{3.20}$$

To this one has to add a term which breaks κ-symmetry and adds kinetic terms for the target-space fermions and coupling to the RR-flux $P^{\alpha\hat{\alpha}}$

$$S_\kappa = \int d^2z (d_\alpha \bar{J}_1^\alpha + \bar{d}_{\hat{\alpha}} J_3^{\hat{\alpha}} + P^{\alpha\hat{\alpha}} d_\alpha \bar{d}_{\hat{\alpha}}) = \int d^2z \langle d\bar{J}_1 - \bar{d} J_3 + d\bar{d} \rangle \,, \tag{3.21}$$

where, in curved backgrounds, the d's are the conjugate variables to the superspace coordinates θ's. After integrating out d and $\bar{d}$ we get the complete matter part

$$S_{\rm GS} + S_\kappa = \int d^2z \langle \frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 + \frac{3}{4} J_3 \bar{J}_1 \rangle \ . \tag{3.22}$$

This sigma-model can be recognized as taking the same form as the sigma-model used in [29] for the compactification of type II superstring on $AdS_2 \times S^2 \times CY_3$ in the hybrid formalism. It is a general fact that the matter part of the hybrid and the pure spinor formalism is the same. As usual this has to be supplemented with kinetic terms for the pure spinors and their coupling to the background

$$S_{gh} = \int d^2z \left\langle w\bar{\partial}\lambda + \bar{w}\partial\bar{\lambda} + N\bar{J}_0 + \bar{N}J_0 - N\bar{N} \right\rangle \tag{3.23}$$

in order to obtain the full superstring sigma-model (3.19) with action $S = S_{GS} + S_\kappa + S_{gh}$.

3.3 Classical integrability of the pure spinor sigma-model

In this subsection we will demonstrate the classical integrability of the action (3.19). For finding the equations of motion and the flat currents we follow the method of [30]. Here, one has to distinguish between two cases — a non-Abelian gauge symmetry H and an Abelian one, which occurs only in the two-dimensional non-critical superstrings. We begin with the non-Abelian case and then discuss the differences when the gauge group is Abelian.

The equations of motion of the currents J_i are obtained by considering the variation $\delta g = gX$ under which $\delta J = \partial X + [J, X]$ and using the $\mathbb{Z}_4$ grading and the Maurer-Cartan equations, so that we get

$$\nabla \bar{J}_3 = -[J_1, \bar{J}_2] - [J_2, \bar{J}_1] + [N, \bar{J}_3] + [\bar{N}, J_3] \,, \tag{3.24}$$
$$\bar{\nabla} J_3 = [N, \bar{J}_3] + [\bar{N}, J_3] \,, \tag{3.25}$$
$$\nabla \bar{J}_2 = -[J_1, \bar{J}_1] + [N, \bar{J}_2] + [\bar{N}, J_2] \,, \tag{3.26}$$
$$\bar{\nabla} J_2 = [J_3, \bar{J}_3] + [N, \bar{J}_2] + [\bar{N}, J_2] \,, \tag{3.27}$$
$$\nabla \bar{J}_1 = [N, \bar{J}_1] + [\bar{N}, J_1] \,, \tag{3.28}$$
$$\bar{\nabla} J_1 = [J_2, \bar{J}_3] + [J_3, \bar{J}_2] + [N, \bar{J}_1] + [\bar{N}, J_1] \,, \tag{3.29}$$

where $\nabla J = \partial J + [J_0, J]$ and $\bar{\nabla} J = \bar{\partial} J + [\bar{J}_0, J]$ are the gauge covariant derivatives. The equations of motion of the pure spinors and the pure spinor gauge currents are

$$\bar{\nabla}\lambda = [\bar{N}, \lambda]\,, \qquad \nabla\bar{\lambda} = [N, \bar{\lambda}]\,, \tag{3.30}$$
$$\bar{\nabla}N = -[N, \bar{N}]\,, \quad \nabla\bar{N} = [N, \bar{N}]\,. \tag{3.31}$$

As in the previous section on the GS formalism, we are looking for a one-parameter family of right-invariant flat currents $a(\mu)$. The left-invariant current $A = g^{-1}ag$ constructed from the flat current a satisfies the equation

$$\nabla\bar{A} - \bar{\nabla}A + [A, \bar{A}] + \sum_{i=1}^{3}\left([J_i, \bar{A}] + [A, \bar{J}_i]\right) = 0\,. \tag{3.32}$$

A and $\bar{A}$ can depend on all the currents for which there are equations of motion so

$$A = c_2 J_2 + c_1 J_1 + c_3 J_3 + c_N N\,, \quad \bar{A} = \bar{c}_2 \bar{J}_2 + \bar{c}_1 \bar{J}_1 + \bar{c}_3 \bar{J}_3 + \bar{c}_N \bar{N}\,. \tag{3.33}$$

By requiring the coefficients of the currents to satisfy (3.32) one obtains the equations

$$\begin{aligned}
&-\bar{c}_2 + c_1\bar{c}_1 + \bar{c}_1 + c_1 = 0\,, && -\bar{c}_3 + c_1\bar{c}_2 + \bar{c}_2 + c_1 = 0 && -\bar{c}_3 + c_2\bar{c}_1 + \bar{c}_1 + c_2 = 0\,,\\
&-c_2 + c_3\bar{c}_3 + \bar{c}_3 + c_3 = 0 && -c_1 + c_2\bar{c}_3 + \bar{c}_3 + c_2 = 0\,, && -c_1 + c_3\bar{c}_2 + \bar{c}_2 + c_3 = 0\,,\\
&\bar{c}_1 - c_1 + c_N\bar{c}_1 + c_N = 0\,, && \bar{c}_1 - c_1 - c_1\bar{c}_N - \bar{c}_N = 0\,, && \bar{c}_2 - c_2 + c_N\bar{c}_2 + c_N = 0\,,\\
&\bar{c}_2 - c_2 - c_2\bar{c}_N - \bar{c}_N = 0 && \bar{c}_3 - c_3 + c_N\bar{c}_3 + c_N = 0\,, && \bar{c}_3 - c_3 - c_3\bar{c}_N - \bar{c}_N = 0\,,\\
&c_2\bar{c}_2 + \bar{c}_2 + c_2 = 0\,, && c_1\bar{c}_3 + \bar{c}_3 + c_1 = 0\,, && c_3\bar{c}_1 + \bar{c}_1 + c_3 = 0\,,\\
&\bar{c}_N + c_N + c_N\bar{c}_N = 0\,,
\end{aligned} \tag{3.34}$$

whose solutions can be written as

$$\begin{aligned}
c_2 &= \mu^{-1} - 1\,, & c_1 &= \pm\mu^{-1/2} - 1\,, & c_3 &= \pm\mu^{-3/2} - 1\,, & \bar{c}_2 &= \mu - 1\,,\\
\bar{c}_1 &= \pm\mu^{3/2} - 1\,, & \bar{c}_3 &= \pm\mu^{1/2} - 1\,, & c_N &= \mu^{-2} - 1\,, & \bar{c}_N &= \mu^2 - 1\,.
\end{aligned} \tag{3.35}$$

Hence, there exists a one-parameter set of flat currents.

The flat currents are given by the right-invariant versions $a = gAg^{-1}$ and $\bar{a} = g\bar{A}g^{-1}$ of the currents A and $\bar{A}$ found above. The conserved charges are given by

$$U_C = \mathrm{P}\exp\left[-\int_C (dza + d\bar{z}\bar{a})\right]\,. \tag{3.36}$$

These charges should be BRST-closed in order to represent physical symmetries. In appendix B.2 it is shown that these charges are indeed BRST invariant.

The construction of the flat currents in the case of an Abelian gauge group is very similar with some differences we will now discuss. The equations of the pure spinor gauge generators (3.31) degenerate in the Abelian case into the equations

$$\bar{\partial}N = 0\,, \quad \partial\bar{N} = 0\,. \tag{3.37}$$

As a result, the last equation in (3.34) drops. However, the solution (3.35) remains valid. The proof of the classical BRST invariance of these charges is identical to the one in the non-Abelian case.

The first two conserved charges can be obtained by expanding $\mu = 1 + \epsilon$ about $\epsilon = 0$. To simplify the notation we will consider the right invariant currents

$$j_i \equiv g J_i g^{-1} , \quad \bar{j}_i \equiv g \bar{J}_i g^{-1} , \quad n \equiv g N g^{-1} , \quad \bar{n} \equiv g \bar{N} g^{-1} . \tag{3.38}$$

Using the expansion in ϵ one gets

$$a = -\left(\frac{1}{2}j_1 + j_2 + \frac{3}{2}j_3 + 2n\right)\epsilon + \left(\frac{3}{8}j_1 + j_2 + \frac{15}{8}j_3 + 3n\right)\epsilon^2 + O(\epsilon^3) , \tag{3.39}$$

$$\bar{a} = \left(\frac{3}{2}\bar{j}_1 + \bar{j}_2 + \frac{1}{2}\bar{j}_3 + 2\bar{n}\right)\epsilon + \left(\frac{3}{8}\bar{j}_1 - \frac{1}{8}\bar{j}_3 + \bar{n}\right)\epsilon^2 + O(\epsilon^3) , \tag{3.40}$$

whose substitution in (3.36) and using $U_C = 1 + \sum_{n=1}^{\infty} \epsilon^n Q_n$ yields

$$Q_1 = \int_C \left[dz \left(\frac{1}{2}j_1 + j_2 + \frac{3}{2}j_3 + 2n\right) - d\bar{z}\left(\frac{3}{2}\bar{j}_1 + \bar{j}_2 + \frac{1}{2}\bar{j}_3 + 2\bar{n}\right)\right] , \tag{3.41}$$

$$\begin{aligned} Q_2 = & -\int_C \left[dz \left(\frac{3}{8}j_1 + j_2 + \frac{15}{8}j_3 + 3n\right) + d\bar{z}\left(\frac{3}{8}\bar{j}_1 - \frac{1}{8}\bar{j}_3 + \bar{n}\right)\right] + \\ & + \int_C \left[dz \left(\frac{1}{2}j_1 + j_2 + \frac{3}{2}j_3 + 2n\right)\Big|_{(z,\bar{z})} - d\bar{z}\left(\frac{3}{2}\bar{j}_1 + \bar{j}_2 + \frac{1}{2}\bar{j}_3 + 2\bar{n}\right)\Big|_{(z,\bar{z})}\right] \times \\ & \times \int_o^{(z,\bar{z})} \left[dz' \left(\frac{1}{2}j_1 + j_2 + \frac{3}{2}j_3 + 2n\right)\Big|_{(z',\bar{z}')} - \right. \\ & \left. - d\bar{z}'\left(\frac{3}{2}\bar{j}_1 + \bar{j}_2 + \frac{1}{2}\bar{j}_3 + 2\bar{n}\right)\Big|_{(z',\bar{z}')}\right] . \end{aligned} \tag{3.42}$$

The first charge Q_1 is the local Noether charge. The rest of the conserved charges, which form the Yangian algebra, can be obtained by repetitive commutators of Q_2.

3.4 Adding D-branes to the pure spinor superstrings

In this section we will consider the addition of D-branes to the coset space background. For this purpose we consider the implications of adding boundaries to the worldsheet (adding D-branes in the pure spinor formalism is treated in [31, 32]) and requiring the appropriate boundary conditions.

The contribution of the boundary to the variation $\delta g = gX$ of the pure spinor action is

$$\begin{aligned} \delta S = i \oint_{\partial\Sigma} \Big\langle & \left(\frac{1}{4}d\bar{z}\bar{J}_3 - \frac{3}{4}dzJ_3\right)X_1 + \frac{1}{2}(d\bar{z}\bar{J}_2 - dzJ_2)X_2 + \left(\frac{3}{4}d\bar{z}\bar{J}_1 - \frac{1}{4}dzJ_1\right)X_3 + \\ & + d\bar{z}\bar{w}\delta\bar{\lambda} - dzw\delta\lambda \Big\rangle + \dots , \end{aligned} \tag{3.43}$$

where X has been decomposed into its $\mathbb{Z}_4$ invariant components X_i and the $\dots$ are the worldsheet bulk terms. We will consider the worldsheet as the upper-half complex plane so the boundary is given by $z = \bar{z}$. The boundary conditions that follow are

$$\bar{J}_3\Big|_{\partial\Sigma} = 3J_3\Big|_{\partial\Sigma}, \quad J_2\Big|_{\partial\Sigma} = \bar{J}_2\Big|_{\partial\Sigma}, \quad J_1\Big|_{\partial\Sigma} = 3\bar{J}_1\Big|_{\partial\Sigma}, \quad w_\alpha \delta\lambda^\alpha\Big|_{\partial\Sigma} = -\bar{w}_{\hat{\alpha}}\delta\bar{\lambda}^{\hat{\alpha}}\Big|_{\partial\Sigma}. \quad (3.44)$$

An additional constraint comes from requiring the action to be BRST invariant. The BRST variation of the action is

$$\delta_B S = \frac{i}{4}\oint_{\partial\Sigma}\langle d\bar{z}\epsilon(\lambda\bar{J}_3 - \bar{\lambda}\bar{J}_1) - dz\epsilon(\bar{\lambda}J_1 - \lambda J_3)\rangle, \quad (3.45)$$

which after substituting (3.44) takes the form

$$\delta_B S = i\oint_{\partial\Sigma}(dz\epsilon j_B - d\bar{z}\epsilon\bar{j}_B), \quad (3.46)$$

so we have to require in addition $j_B = \bar{j}_B$ on the boundary.

We may solve the pure spinor boundary conditions by

$$(\lambda^\alpha - R^\alpha{}_{\hat{\alpha}}\bar{\lambda}^{\hat{\alpha}})\Big|_{\partial\Sigma} = 0, \quad (w_\alpha + R_\alpha{}^{\hat{\alpha}}\bar{w}_{\hat{\alpha}})\Big|_{\partial\Sigma} = 0, \quad (3.47)$$

in which the matrix $R_\alpha{}^{\hat{\alpha}}$ determines the type of D-brane and $R_\alpha{}^{\hat{\alpha}}R^\alpha{}_{\hat{\beta}} = \delta^{\hat{\alpha}}_{\hat{\beta}}$. The BRST boundary condition then becomes

$$(\bar{J}_1^\alpha - R^\beta{}_{\hat{\beta}}\eta_{\beta\hat{\alpha}}\eta^{\alpha\hat{\beta}}J_3^{\hat{\alpha}})\Big|_{\partial\Sigma} = 0. \quad (3.48)$$

This condition can also be obtained by requiring that the boundary condition involving w and $\bar{w}$ be BRST invariant.

The matrix R is to be determined by the symmetries that the D-brane configuration breaks. However, since the boundary conditions for the matter fields (3.44) involve only left-invariant currents, they alone are not sufficient in order to break some of the symmetries. In order to gain such information it is probably necessary to resort to a specific parameterization of the super-Lie manifold G and the gauged subgroup H.

4. Quantum consistency of the pure spinor sigma-model

In this section we will show that the pure spinor superstring on the supercoset backgrounds in various dimensions is gauge invariant and BRST invariant to all orders in the sigma-model perturbation theory. Then, we will show that the infinite set of nonlocal charges, which are classically conserved, are also BRST invariant in the quantum theory, proving that the integrability of the superstring holds quantum mechanically as well.

Since our backgrounds are realized in terms of supercosets with a $\mathbb{Z}_4$ automorphism, we will be able to apply the powerful tools developed in [10] for the superstring on the $AdS_5 \times S^5$ background. The only subtlety is related to the different definitions of the pure spinor constraints in the lower dimensional cases.

4.1 Quantum gauge invariance

As we discussed above, the action is classically gauge invariant under the right multiplication $g \to gh$, where $h \in H$. We will prove that we can always add a local counterterm such that the quantum effective action remains gauge invariant at the quantum level.[13] Quantum gauge invariance will then be used to prove BRST invariance.

An anomaly in the H gauge invariance would show up as a nonvanishing gauge variation of the effective action $\delta_\Lambda S_{\rm eff}$ in the form of a local operator. Since there is no anomaly in the global H invariance, the variation must vanish when the gauge parameter is constant and, moreover, it must have grading zero. Looking at the list of our worldsheet operators, we find that the most general form of the variation is

$$\delta S_{\rm eff} = \int d^2z \langle c_1 N \bar{\partial}\Lambda + \bar{c}_1 \bar{N} \partial\Lambda + 2c_2 J_0 \bar{\partial}\Lambda + 2\bar{c}_2 \bar{J}_0 \partial\Lambda \rangle, \tag{4.1}$$

where $\Lambda = T_{[ab]}\Lambda^{[ab]}(z,\bar{z})$ is the local gauge parameter and $(c_1, \bar{c}_1, c_2, \bar{c}_2)$ are arbitrary coefficients. By adding the counterterm

$$S_c = -\int d^2z \langle c_1 N \bar{J}_0 + \bar{c}_1 \bar{N} J_0 + (c_2 + \bar{c}_2) J_0 \bar{J}_0 \rangle, \tag{4.2}$$

we find that the total variation becomes

$$\delta_\Lambda (S_{\rm eff} + S_c) = (c_2 - \bar{c}_2) \int d^2z \langle J_0 \bar{\partial}\Lambda - \bar{J}_0 \partial\Lambda \rangle. \tag{4.3}$$

On the other hand, the consistency condition on the gauge anomaly requires that

$$(\delta_\Lambda \delta_{\Lambda'} - \delta_{\Lambda'} \delta_\Lambda) S_{\rm eff} = \delta_{[\Lambda,\Lambda']} S_{\rm eff}, \tag{4.4}$$

which fixes the coefficients $c_2 = \bar{c}_2$. Therefore the action is gauge invariant quantum mechanically.

4.2 Quantum BRST invariance

In order to prove the BRST invariance of the superstring at all orders in perturbation theory we will adapt the proof of [10] to our lower-dimensional cases. First, we will show that the classical BRST charge is nilpotent. We will then prove that the effective action can be made classically BRST invariant by adding a local counterterm, using triviality of a classical cohomology class. Then we will prove that order by order in perturbation theory no anomaly in the BRST invariance can appear.

As we have shown in the previous section, the action (3.19) in the pure spinor formalism is classically BRST invariant. It is easy to prove, following the algebraic argument [10], that, in all our backgrounds, the pure spinor BRST charge is classically nilpotent on the pure spinor constraint, up to gauge invariance and the ghost equations of motion. The second variation of the ghost currents reads indeed

$$\begin{aligned} Q^2(N) &= -[N,\Lambda] - \{\lambda, \nabla\bar{\lambda} - [N,\bar{\lambda}]\}, \\ Q^2(\bar{N}) &= -[\bar{N},\Lambda] - \{\bar{\lambda}, \bar{\nabla}\lambda - [\bar{N},\lambda]\}, \end{aligned} \tag{4.5}$$

[13]This proof is different from the one in [10]. In that paper, Berkovits uses a parity symmetry argument, while we use the consistency condition on gauge anomalies.

for the particular gauge transformation parameterized by $\Lambda = \{\lambda, \bar{\lambda}\}$ and the equations of motion (3.31). Therefore the classical BRST charge is well defined.

Consider now the quantum effective action $S_{\rm eff}$. After the addition of a suitable counterterm, it is gauge invariant to all orders. Moreover, the classical BRST transformations of (3.18) commute with the gauge transformations, since the BRST charge is gauge invariant. Therefore, the anomaly in the variation of the effective action, which is a local operator, must be a gauge invariant integrated vertex operator of ghost number one

$$\delta_{BRST} S_{\rm eff} = \int d^2z \langle \Omega^{(1)}_{z\bar{z}} \rangle. \tag{4.6}$$

In appendix B.3 we show that the cohomology of such operators is empty, namely that we can add a local counterterm to cancel the BRST variation of the action. A crucial step in the proof is that the symmetric bispinor, constructed with the product of two pure spinors, is proportional to the middle dimensional form. Schematically, this means that in $d = 2n$ dimensions we can decompose

$$\lambda^\alpha \lambda^\beta \sim \sigma^{\alpha\beta}_{m_1 \ldots m_n} (\lambda \sigma^{m_1 \ldots m_n} \lambda).$$

In section 3.1, we have shown that this property is satisfied by the pure spinors in all our backgrounds, ensuring classical BRST invariance of the effective action.

Since there are no conserved currents of ghost number two in the cohomology that could deform Q^2, the quantum modifications to the BRST charge can be chosen such that its nilpotence is preserved. In this case, we can set the anti-fields to zero and use algebraic methods to extend the BRST invariance of the effective action by induction to all orders in perturbation theory. Suppose the effective action is invariant to order h^{n-1}. This means that

$$\widetilde{Q} S_{\rm eff} = h^n \int d^2z \langle \Omega^{(1)}_{z\bar{z}} \rangle + \mathcal{O}(h^{n+1}).$$

The quantum modified BRST operator $\widetilde{Q} = Q + Q_q$ is still nilpotent up to the equations of motion and the gauge invariance. This implies that $Q \int d^2z \langle \Omega^{(1)}_{z\bar{z}} \rangle = 0$. But the cohomology of ghost number one integrated vertex operators is empty, so $\Omega^{(1)}_{z\bar{z}} = Q \Sigma^{(0)}_{z\bar{z}}$, which implies

$$\widetilde{Q} \left(S_{\rm eff} - h^n \int d^2z \langle \Sigma^{(0)}_{z\bar{z}} \rangle \right) = \mathcal{O}(h^{n+1}). \tag{4.7}$$

Therefore, order by order in perturbation theory it is possible to add a counterterm that restores BRST invariance.

4.3 Quantum integrability

In this subsection we will finally show that the classically conserved nonlocal currents of (3.35) can be made BRST invariant quantum mechanically. In this way we prove quantum integrability of our type II superstring theories. The proof is essentially identical to the one presented in [10]. First, we review how the absence of a certain ghost number two state from the cohomology implies the existence of an infinite number of nonlocal

BRST invariant charges. Then we will review how this argument can be extended quantum mechanically.

Consider the charge that generates the global symmetry with respect to the supergroup G

$$q \equiv q^A T_A = \int d\sigma j^A T_A, \tag{4.8}$$

where j^A is the corresponding gauge invariant current. Since this is a symmetry of the theory, the charge is BRST invariant, so we find $\epsilon Q j = \partial_\sigma h$, where $h = h^A T_A$ is a certain operator of ghost number one and weight zero. Classical nilpotence of the BRST charge implies moreover that $Qh = 0$.

Consider now the operator $:\{h, h\}:$, where $:\ldots:$ denotes a BRST invariant normal ordering prescription. If there exists a ghost number one and weight zero operator Ω, such that

$$Q\Omega =: \{h, h\} :, \tag{4.9}$$

then there is an infinite number of nonlocal charges which are classically BRST invariant. To prove this, consider the nonlocal operator

$$k =: \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{\sigma} d\sigma' [j(\sigma), j(\sigma')] : . \tag{4.10}$$

Its BRST variation is $Qk = 2 : \int_{-\infty}^{+\infty} d\sigma [j(\sigma), h(\sigma)] :$. On the other hand, the BRST transformations are classically nilpotent, in fact we find $Q(2 : [j(\sigma), h(\sigma)] :) = \partial_\sigma : \{h(\sigma), h(\sigma)\} :$. Now, since there is an operator Ω that satisfies (4.9), we have

$$Q(2 : [j, h] : -\partial_\sigma \Omega) = 0. \tag{4.11}$$

In other words, the ghost number one weight one operator $2 : [j, h] : -\partial_\sigma \Omega$ is BRST closed. On the other hand, the BRST cohomology of ghost number one currents $\mathcal{O}^{(1)}_\sigma$ is empty, as we will show below. We conclude that this operator is BRST exact, namely there exists a $\Sigma^{(0)}$ such that $Q\Sigma^{(0)} = 2 : [j, h] : -\partial_\sigma \Omega$. But then the nonlocal charge

$$\widetilde{q} = k - \int_{-\infty}^{+\infty} d\sigma \Sigma, \tag{4.12}$$

is classically BRST invariant and represent the first nonlocal charge of the Yangian. By commuting $\widetilde{q}$ with itself one generates the whole Yangian.

It remains to be shown that the BRST cohomology of ghost number one currents is trivial. This cohomology, in fact, is equivalent to the cohomology of ghost number two unintegrated vertex operators, by the usual descent relation

$$Q \int d\sigma \mathcal{O}^{(1)}_\sigma = 0 \Rightarrow Q\mathcal{O}^{(1)}_\sigma = \partial_\sigma \mathcal{O}^{(2)}. \tag{4.13}$$

At ghost number two we have only two unintegrated vertex operators that transform in the adjoint of the global supergroup G, namely

$$V_1 = g\lambda\bar{\lambda}g^{-1}, \qquad V_2 = g\bar{\lambda}\lambda g^{-1}. \tag{4.14}$$

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Their sum is BRST closed, while their difference is not. Finally, we have $V_1 + V_2 = Q\Omega^{(1)}$ where

$$\Omega^{(1)} = \frac{1}{2} g(\lambda + \bar{\lambda}) g^{-1}, \tag{4.15}$$

so this classical cohomology class is empty.

Now, suppose that we have a BRST invariant nonlocal charge q at order h^{n-1} in perturbation theory, namely $\widetilde{Q}q = h^n \Omega^{(1)} + \mathcal{O}(h^{n+1})$. $\Omega^{(1)}$ must be a ghost number one local charge, since any anomaly must be proportional to a local operator. Nilpotence of the quantum BRST charge $\widetilde{Q} = Q + Q_q$ implies that $Q\Omega^{(1)} = 0$, but the classical cohomology at ghost number one and weight one is empty, as shown above, so there exists a current $\Sigma^{(0)}(\sigma)$ such that $Q \int d\sigma \Sigma^{(0)}(\sigma) = \Omega^{(1)}$. As a result $\widetilde{Q}(q - h^n \int d\sigma \Sigma^{(0)}(\sigma)) = \mathcal{O}(h^{n+1})$. Hence, we have shown that it is possible to modify the classically BRST invariant charges of (3.35) such that they remain BRST invariant at all orders in perturbation theory.

5. One-loop conformal invariance

In this section we will give the spacetime interpretation of the various sigma-models we have introduced in the previous sections. Some of these backgrounds describe the noncompact part of a ten-dimensional critical superstring, while some others describe lower-dimensional non-critical superstrings. The way we will identify the correct superstring is by looking at the Ricci scalar of the backgrounds, which vanishes for the backgrounds being a part of a compactification.

The coefficients of the one-loop beta-function equations for the conformal invariance of a sigma-model on a supercoset G/H with $\mathbb{Z}_4$ automorphism are proportional to the super Ricci tensor of the supergroup G. This has been shown for the matter part of the hybrid formalism in [33, 29] (which is identical to the matter part of the pure spinor action) and we will show below that the same holds for the ghost part of the pure spinor action. Whenever the supergroup G is super Ricci flat (its dual Coxeter number vanishes), the sigma-model is automatically conformally invariant at one-loop. The supercosets describing $AdS_p \times S^p$ backgrounds are all super Ricci flat and therefore conformally invariant. Moreover, since the AdS_p and the S^p part have the same radii, their scalar curvatures have equal modulus but opposite sign and hence the total scalar curvature of the background vanishes. Since in these backgrounds the dilaton is constant and the scalar curvature vanishes, the Weyl anomaly also vanishes and they necessarily describe a part of a critical ten-dimensional background. The compactified part has to be Ricci flat and preserve minimal supersymmetry, hence a CY manifold of complex dimension $5 - p$ would do the job and we can identify the full ten-dimensional background as $AdS_p \times S^p \times CY_{5-p}$.

When the supergroup G is not super Ricci flat (its dual Coxeter number is nonvanishing), the GS sigma-model can be still shown to be conformal at one-loop, as first discussed by Polyakov [3]. The intuitive reason, which we will explain below, is that classical κ-symmetry of the GS action, which is responsible for spacetime supersymmetry, is enough to ensure one-loop conformal invariance. The scalar curvature of these backgrounds is nonvanishing and we will argue that they describe a non-critical superstring, along the lines of [3]. However, we will not compute the Weyl beta-function, in other words we are not computing the central charge. As a heuristic check of consistency, we will just show that in all cases the naive central charge of the matter plus ghost action that we get in the free field theory limit vanishes. However, a precise computation of the central charge for the non-critical case would require the ability to analyze strongly coupled sigma-models.

Before going into the details, let us summarize the results. We will collect first the ten-dimensional backgrounds and then the non-critical ones. In all cases we have given both the classical Green-Schwarz and the quantum pure spinor sigma models.[14]

Critical superstrings

The following backgrounds are interpreted as the non-compact part of a ten-dimensional type II background $AdS_p \times S^p \times CY_{5-p}$.

- $AdS_2 \times S^2$ with RR two-form flux, realized as
$$AdS_2 \times S^2 : \frac{PSU(1,1|2)}{U(1)\times U(1)}, \tag{5.1}$$
is super Ricci flat. Therefore, it is the non-compact part of the ten-dimensional type IIA background obtained by tensoring it with a compact CY threefold as in [29].
- $AdS_3 \times S^3$ with RR three-form flux, realized as
$$AdS_3 \times S^3 : \frac{PSU(1,1|2)^2}{\mathrm{SO}(1,2)\times \mathrm{SO}(3)}, \tag{5.2}$$
is super Ricci flat as well. Therefore, it is the noncompact part of the ten-dimensional type IIB background obtained by tensoring it with a compact CY twofold.

Non-critical superstrings

The following backgrounds are interpreted as non-critical superstrings. The AdS_{2n} backgrounds with $2n$ units of RR-flux, which we realized as
$$\begin{aligned} AdS_2 &: \tfrac{Osp(2|2)}{\mathrm{SO}(1,1)\times \mathrm{SO}(2)} \\ AdS_4 &: \tfrac{Osp(2|4)}{\mathrm{SO}(1,3)\times \mathrm{SO}(2)} \\ AdS_6 &: \tfrac{F(4)}{\mathrm{SO}(1,5)\times \mathrm{SL}(2)} \end{aligned} \tag{5.3}$$
describe type IIA non-critical superstrings in $2n$ dimensions.[15]

[14] While the GS sigma-model always describes the full superstring, some subtleties concern the pure spinor action. In this latter case, it has been argued in [11] that the non-critical pure spinor formalism describes the full non-critical superstring spectrum. On the other hand, it might be that the $AdS_p \times S^p \times CY_{5-p}$ lower-dimensional pure spinor action are to be interpreted as the "topological sector" of the full ten-dimensional superstring compactified on CY. The reason for this is that the cohomology of lower-dimensional pure spinor theories in flat Minkowski describes the off-shell multiplets of lower-dimensional supersymmetry [24, 34], and the same structure might carry on to other curved backgrounds.

[15] In addition, the AdS_2 non-critical background can be realized as an $Osp(1|2)/\mathrm{SO}(2)$ supercoset. The classical GS sigma-model for this supercoset is well defined [22]. However, as we will see, in its quantum realization as a pure spinor superstring all the correlation functions vanish. We do not know how to interpret this fact.

5.1 One-loop beta-function

Before verifying that the various backgrounds are conformally invariant at one-loop, we would like to warn the reader about the validity of such computations.

The backgrounds of the ten-dimensional critical superstring can be usually considered in the regime in which the spacetime curvature is very small. In this regime supergravity is a good approximation. In the example of $AdS_5 \times S^5$, this is the limit where the radius of AdS is very large. Since the radius corresponds to the inverse coupling of the sigma-model, the small curvature limit is realized as the weak coupling regime of the sigma-model. Thus, in this case it makes sense to study the conformal invariance of the worldsheet theory order by order in the sigma-model perturbation theory and one finds that one-loop conformal invariance requires an on-shell supergravity background (small curvature limit). Higher loops in the sigma-model describe higher curvature corrections to the supergravity equations of motion.

In the case of non-critical superstrings things are typically different. Namely, the curvature is always at string scale. In fact, as we already mentioned, there is no regime in which non-critical supergravity (one-loop perturbation theory in the sigma-model) provides a reliable description of the spacetime.[16] Therefore the sigma-models that we described in the previous sections are typically strongly coupled two-dimensional field theories. In particular, they are understood to be living at a fixed point of the worldsheet RG flow.

With this caveat in mind, in this section we will check that these sigma-models are conformally invariant at one-loop. We take this as an evidence for the existence of these theories, while we leave for a future analysis a proof of conformal invariance at all orders in perturbation theory.

We review the computation of the one-loop conformal beta-function in the GS sigma-models [3]. We will not consider the one-loop beta-function for the Weyl anomaly. We will see then that the contribution of the bosonic part to the one-loop effective action precisely cancels the contribution of the fermionic part, proving one-loop conformal invariance. This is due to the fact that κ-symmetry fixes the number of physical bosons equal to the number of physical fermions, implementing therefore spacetime supersymmetry. A sigma-model on a d-dimensional background has $d-2$ physical bosonic degrees of freedom in both left and right moving sectors. κ-symmetry requires that the number of physical fermions should also be $d-2$ in both the left and right moving sectors, which fixes the total number of real spacetime supersymmetries to $4(d-2)$. This gives us sixteen supersymmetries in six dimensions and eight supersymmetries in four dimensions (which is the same number as required in type II compactification on CY). In two dimensions, however, since κ-symmetry removes all the fermionic degrees of freedom, we can have more possibilities, namely two or four. We will argue in the next section what happens in this last case.

Here we review the computation of the one-loop beta-function of the AdS_4 coset $\frac{OSp(2|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(2)}$ performed in [3] adapting it to the notations used in this paper. We be-

[16] By non-critical supergravity, as will be clarified below, we mean lower-dimensional supergravity with a cosmological constant term, fixed at string scale value.

gin with the action (2.10) which in the conformal gauge reads

$$S = \frac{1}{\lambda^2} \int d^2\sigma \left\langle \frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 - \frac{1}{4} J_3 \bar{J}_1 \right\rangle , \tag{5.4}$$

in which the coupling λ is written explicitly. The dependence of the sigma-model coupling λ on the string coupling g_S and the RR flux N_c is given by

$$\frac{1}{\lambda^2} = g_S N_c \ . \tag{5.5}$$

We consider the quantum fluctuations $X \in osp(2|4)$ around the classical background $\widetilde{g}$ such that $g = \widetilde{g} e^{\lambda X}$. The currents are given by

$$J = e^{-\lambda X} \partial e^{\lambda X} + e^{-\lambda X} \widetilde{J} e^{\lambda X} , \tag{5.6}$$

and the corresponding equation for the right-moving current, where $\widetilde{J} = \widetilde{g}^{-1} \partial \widetilde{g}$ and similarly for the right-movers. As argued in [29] the gauge $X \in \mathcal{G} \backslash \mathcal{H}_0$ can be chosen. The one-loop beta-function is obtained from the second order expansion in λ of the action. By computing the first order expansion, integrating by parts and making use of the Maurer-Cartan equations to express the derivatives of the currents J_1 and J_3 in terms of the commutators of currents one gets

$$\begin{aligned} S_1 = \frac{1}{\lambda} \int d^2z \Big\langle & \frac{1}{2} \partial X_2 \bar{J}_2 + \frac{1}{2} J_2 \bar{\partial} X_2 - \frac{1}{2} ([J_0, \bar{J}_2] - [J_2, \bar{J}_0] - [J_1, \bar{J}_1] + [J_3, \bar{J}_3]) X_2 + \\ & + [J_2, \bar{J}_1] X_1 - [J_3, \bar{J}_2] X_3 \Big\rangle , \end{aligned} \tag{5.7}$$

where we dropped the tilde on the background currents for simplicity of notation. The second variation is then computed and as in [3] it is convenient to restrict the computations to backgrounds with $J_1 = J_3 = \bar{J}_1 = \bar{J}_3 = 0$ since κ-symmetry guarantees that the beta-function associated with the other terms in the action will be equal to the beta-function of the term $J_2 \bar{J}_2$. In such backgrounds the action for the X fields reduces to

$$\begin{aligned} S_X = \int d^2z \Big\langle & \partial X_2 \bar{\partial} X_2 + [\bar{J}_0, X_2] \partial X_2 + [J_0, X_2] \bar{\partial} X_2 - [\bar{J}_2, X_2][J_2, X_2] + [\bar{J}_0, X_2][J_0, X_2] - \\ & - [J_2, X_1] \bar{\partial} X_1 - [\bar{J}_2, X_3] \partial X_3 - [J_2, X_1][\bar{J}_0, X_1] - 2[J_2, X_1][\bar{J}_2, X_3] - \\ & - [\bar{J}_2, X_3][J_0, X_3] + O(\lambda) \Big\rangle . \end{aligned} \tag{5.8}$$

In order to compute the one-loop quantum corrections to the $J_2 \bar{J}_2$ term we write the relevant parts of the action in terms of the structure constants and the Cartan metric

$$\begin{aligned} S_X = \int d^2z (& \eta_{ab} \partial X_2^a \bar{\partial} X_2^b + \eta_{[ef][gh]} f_{ab}^{[ef]} f_{cd}^{[gh]} J_2^a \bar{J}_2^c X_2^b X_2^d + \eta_{\hat{\alpha}\alpha} f_{a\beta}^{\hat{\alpha}} J_2^a X_1^\beta \bar{\partial} X_1^\alpha + \\ & + \eta_{\alpha\hat{\alpha}} f_{a\hat{\beta}}^{\alpha} \bar{J}_2^a X_3^{\hat{\beta}} \partial X_3^{\hat{\alpha}} + 2\eta_{\hat{\alpha}\alpha} f_{a\beta}^{\hat{\alpha}} f_{b\hat{\beta}}^{\alpha} J_2^a \bar{J}_2^b X_1^\beta X_3^{\hat{\beta}} + \dots) . \end{aligned} \tag{5.9}$$

Upon substituting the structure constants of $OSp(2|4)$ (see appendix C) we get

$$\begin{aligned} S_X = \int d^2z [& \eta_{ab} \partial X_2^a \bar{\partial} X_2^b - (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) J_2^a \bar{J}_2^c X_2^b X_2^d - J_2^a \bar{X}_{1\beta} (\gamma_a)^\beta{}_\alpha \bar{\partial} X_1^\alpha - \\ & - \bar{J}_2^a \bar{X}_{3\hat{\beta}} (\gamma_a)^{\hat{\beta}}{}_{\hat{\alpha}} \partial X_3^{\hat{\alpha}} - J_2^a \bar{J}_2^b \bar{X}_{1\beta} (\gamma_a \gamma_b)^\beta{}_{\hat{\beta}} X_3^{\hat{\beta}} + \dots] , \end{aligned} \tag{5.10}$$

where X_1 and X_3 satisfy the Majorana conditions $\bar{X}_{1\beta} = X_1^\alpha C_{\alpha\beta}$ and $\bar{X}_{3\hat{\beta}} = X_3^{\hat{\alpha}} C_{\hat{\alpha}\hat{\beta}}$. The κ-symmetry is most conveniently fixed as in [3]

$$J_2^a \gamma_a = \sqrt{J_2^a \bar{J}_2^a}\gamma_+ , \quad \bar{J}_2^a \gamma_a = \sqrt{J_2^a \bar{J}_2^a}\gamma_- , \quad X_{1,3} = (J_2^a \bar{J}_2^a)^{-1/4} Y_{1,3} ,$$

where after fixing the residual conformal symmetry the bosonic indices run only on the two transverse directions and the Majorana spinors $Y_{1,3}$ satisfy the light-cone constraints $\gamma_+ Y_3 = \gamma_- Y_1 = 0$. The free field OPEs for the fluctuations now take the form

$$X_2^a(z,\bar{z}) X_2^b(0,0) \sim -\frac{1}{4\pi}\eta^{ab} \log |z|^2 , \tag{5.11}$$

$$\bar{Y}_{1\alpha}(z,\bar{z}) Y_1^\beta(0,0) \sim -\frac{1}{8\pi z}(\gamma_-)^\beta{}_\alpha , \tag{5.12}$$

$$\bar{Y}_{3\hat{\alpha}}(z,\bar{z}) Y_3^{\hat{\beta}}(0,0) \sim -\frac{1}{8\pi \bar{z}}(\gamma_+)^{\hat{\beta}}{}_{\hat{\alpha}} . \tag{5.13}$$

Figure 1: The bosonic (a) and the fermionic (b) diagrams contributing to the one-loop beta-function.

Thus the bosonic one-loop correction to the effective action coming from figure 1(a) is

$$-\frac{1}{2\pi} J_2^a \bar{J}_2^a \log \frac{\Lambda}{\mu} ,$$

where the UV cut-off is $|z| = 1/\Lambda$ and the IR one is $|z| = 1/\mu$. Similarly, the one-loop fermionic correction to the $\langle J_2 \bar{J}_2 \rangle$ comes from the diagram in figure 1(b) and evaluates to

$$\frac{1}{2\pi} J_2^a \bar{J}_2^a \log \frac{\Lambda}{\mu}$$

so the total one-loop correction to $\langle J_2 \bar{J}_2 \rangle$ vanishes. The gauge symmetry of the sigma-model guarantees that no terms involving J_0 appear and unless κ-symmetry does not hold quantum mechanically, the $\langle J_1 \bar{J}_3 \rangle$ and $\langle J_3 \bar{J}_1 \rangle$ are not corrected as well to one-loop.

A note on the difference between the $AdS_p \times S^p$ background and the AdS_p backgrounds is in order. For the former it was found that super-Ricci flatness of the group G in the coset G/H was a sufficient condition for one-loop conformal invariance of $AdS_p \times S^p$ backgrounds [29]. However, as we see here, it is not a necessary condition as demonstrated by the latter case since the Maurer-Cartan equations allow to relate contributions to the $J_1 \bar{J}_3$ and $J_3 \bar{J}_1$ beta-functions to the $J_2 \bar{J}_2$ one leading to the vanishing of the beta-function.

In the AdS_2 case perturbative conformal invariance is trivial because as discussed in subsection 6.1 there are no propagating degrees of freedom after fixing the κ-symmetry.

Pure spinor beta-functions. Let us comment on the computation of the beta-function in the pure spinor formalism in the background field method. Unlike the light-cone GS formalism, we work covariantly at all stages. The matter part of the action is identical to the corresponding formulation of the hybrid superstring on a supercoset with $\mathbb{Z}_4$ automorphism, which was considered in [29]. However, when the supergroups G have nonzero dual Coxeter number, as in the non-critical backgrounds, the various terms rearrange differently.

The contribution to the one-loop effective action coming from the pure spinor sector was considered in [35] for $AdS_5 \times S^5$. It consists of two terms. The first term is obtained by expanding the ghost action $\frac{1}{\lambda^2}\int d^2z \, \langle N\bar{J}_0 + \bar{N}J_0\rangle$ to the second order in the fluctuations of the gauge current J_0. The trilinear couplings

$$\int d^2z \, \langle \widetilde{N}\left([\bar{\partial}X_2, X_2] + [\bar{\partial}X_1, X_3] + [\bar{\partial}X_3, X_1]\right) \tag{5.14}$$

$$+\widetilde{\bar{N}}\left([\partial X_2, X_2] + [\partial X_1, X_3] + [\partial X_3, X_1]\right)\rangle, \tag{5.15}$$

generate the term $\langle \widetilde{N}\widetilde{\bar{N}}\rangle$ in the action through the fish diagram in figure 1(b)

$$\frac{1}{8\pi}\log\frac{\Lambda}{\mu}\widetilde{N}^{[ij]}\widetilde{\bar{N}}^{[kl]}\left(4R_{[ij][kl]}(G) - 4R_{[ij][kl]}(H)\right). \tag{5.16}$$

As explained in [35], there is a second contribution to the one-loop effective action in the ghost sector, coming from the operator $\mathcal{O}(z,\bar{z}) = \langle N\bar{N}\rangle$, which couples the pure spinor Lorentz currents to the spacetime Riemann tensor. The marginal part of the OPE of $\mathcal{O}$ with itself generates at one-loop the following contribution to the effective action

$$\frac{1}{4\pi}\int d^2z \int d^2w \langle \mathcal{O}(z,\bar{z})\mathcal{O}(w,\bar{w})\rangle = \frac{1}{2\pi}\log\frac{\Lambda}{\mu}R_{[ij][kl]}(H)\int d^2z \widetilde{N}^{[ij]}\widetilde{\bar{N}}^{[kl]}, \tag{5.17}$$

which cancels the term proportional to $R_{[ij][kl]}(H)$ in (5.16). So we are left with the following ghost contribution to the one-loop effective action in the ghost sector

$$\frac{1}{2\pi}\log\frac{\Lambda}{\mu}\widetilde{N}^{[ij]}\widetilde{\bar{N}}^{[kl]}R_{[ij][kl]}(G), \tag{5.18}$$

where the explicit expression of the super Ricci tensor of the supergroup in terms of the structure constants is explained in the appendix. In the $AdS_p \times S^p$ cases [29], in which the supergroup G is super Ricci flat, each coupling in the effective action vanishes by itself, all of them being separately proportional to the dual Coxeter number of the supergroup G. However, in the non-critical superstrings, in which the dual Coxeter number of G is nonzero, even if the single terms do not vanish separately, one expects that by making use of Ward identities they give a total vanishing contribution. We just mention that the nontrivial cancellation between the various couplings in the effective action is precisely what happens in the GS computation above. In that case, in the physical gauge there are no ghosts. The bosonic and fermionic part of the beta-function are both non-vanishing (if the dual Coxeter number of G is non-vanishing), however they exactly cancel due to κ-symmetry and using the Maurer-Cartan equations. In the pure spinor formulation, the BRST symmetry plays the role of the κ-symmetry and we have the ghost contribution as well because we work covariantly. At the end of the day, the physical reason for the vanishing of the beta-function would be again spacetime supersymmetry. We leave the proof of one-loop conformal invariance of the pure spinor action in the case of nonzero dual Coxeter number for a future analysis.

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6. The various backgrounds

In this section we give some details on the various backgrounds for which our general construction can be applied. They are all realized as supercosets G/H, where the gauge symmetry H is the invariant locus of a $\mathbb{Z}_4$ automorphism of G. The details of the supergroups and their structure constants are collected in appendix C.

The existing literature on type II superstrings on AdS backgrounds with RR flux, realized as sigma-models on supercosets, is vast. The Green-Schwarz superstring on $AdS_p \times S^p$ has been first constructed in the case $p = 5$ [1] and subsequently in the compactified cases $p = 3$ [36, 37] and $p = 2$ [38]. The Green-Schwarz non-critical superstring on AdS_2 has been proposed in [22],[17] while the non-critical AdS_4 has been discussed in [3]. The type II pure spinor action for $AdS_5 \times S^5$ has been introduced in [23, 39]. The hybrid formalism for the critical cases $p = 2, 3$, whose matter part is similar to the matter part of our pure spinor sigma-models, have been discussed in [33, 29]. The type II pure spinor action for non-critical AdS_4 has been proposed in [11]. The analysis of conformal invariance of such superstring sigma-models in the Green-Schwarz [3], hybrid [33, 29] and pure spinor [35, 10] formulations has received some attention as well.

The proof of the classical integrability of the Green-Schwarz sigma-model on $AdS_5 \times S^5$ [2] has boosted the attention on the integrability of sigma-models on supercosets [7, 6, 9, 8]. Classical integrability of the type II Green-Schwarz superstring on various critical and non-critical backgrounds has been further studied in [3, 5, 4]. The integrability of the pure spinor superstring on $AdS_5 \times S^5$ has been proven first from the classical point of view [30, 21] and afterwards quantum mechanically [10].

6.1 Non-critical AdS_2

The type IIA non-critical superstring on AdS_2 with RR two-form flux is realized as the supercoset $Osp(2|2)/\,\mathrm{SO}(1,1) \times \mathrm{SO}(2)$. The $Osp(2|2)$ supergroup has four bosonic generators $(\mathbf{E}^{\pm}, \mathbf{H}, \widetilde{\mathbf{H}})$ and four fermionic ones $(\mathbf{Q}_\alpha, \mathbf{Q}_{\hat\alpha})$. The index $a = \pm$ denotes the spacetime light-cone directions. The supercharges are real two-dimensional MW spinors, the index $\alpha = 1, 2$ counts the ones with left spacetime chirality and the index $\hat\alpha = \hat 1, \hat 2$ counts the ones with right spacetime chirality (note that in the two-dimensional superstring $\alpha, \hat\alpha$ are not spinor indices but just count the multiplicity of spinors with the same chirality). To obtain AdS_2, we quotient by $\mathbf{H}$ and $\widetilde{\mathbf{H}}$, which generate respectively the SO(1,1) and SO(2) transformations. The $Osp(2|2)$ superalgebra and structure constants are listed in an appendix. The left invariant form $J = G^{-1}dG$ is expanded according to the grading as

$$J_0 = J^H \mathbf{H} + J^{\widetilde H} \widetilde{\mathbf{H}}, \quad J_1 = J^\alpha \mathbf{Q}_\alpha, \quad J_2 = J^a \mathbf{E}_a, \quad J_3 = J^{\hat\alpha} \mathbf{Q}_{\hat\alpha}. \tag{6.1}$$

and the definition of the supertrace is

$$\langle \mathbf{E}_a \mathbf{E}_b \rangle = \delta_a^+ \delta_b^- + \delta_a^- \delta_b^+, \qquad \langle \mathbf{Q}_\alpha \mathbf{Q}_{\hat\alpha} \rangle = \delta_{\alpha\hat\alpha}, \tag{6.2}$$

whose details are given in an appendix.

[17] As explained below, the superstring based on the $Osp(1|2)$ supercoset in [22] is different from ours, based on the $Osp(2|2)$ supercoset.

κ-symmetry of the GS action in two dimensions. The Green-Schwarz sigma-model on AdS_2 is given by (2.10). Let us discuss its κ-symmetry. If we want to define a classical Green-Schwarz superstring in a flat background, it is well known that only in $d = 3, 4, 6, 10$ does κ-symmetry exist. This is due to the existence of the Fierz identities in the gamma matrix algebra, which are needed to define the WZ term. Even if the GS superstring does not exist in flat two dimensions, it does exist on a two-dimensional AdS background with RR two-form flux. The RR flux makes it possible to construct a WZ term, as we showed above. An example of these two-dimensional AdS sigma-models has been discussed in [22]. There is, however, a substantial difference between the usual higher-dimensional κ-symmetry and the two-dimensional one. In higher dimensions this gauge symmetry is reducible. As a result, it removes only half of the θ's from the classical spectrum. In two dimensions, instead, it is not reducible and it removes all the θ's. This fact is expected from two-dimensional on-shell supersymmetry. In the light-cone gauge the two bosonic coordinates are removed from the spectrum, leaving no bosonic degrees of freedom. It is a necessary requirement then that the worldsheet fermionic symmetry remove all the fermionic degrees of freedom as well, and not just half as in higher dimensions. Let us see how this works in detail. We will first briefly review the reducibility of the ten-dimensional κ-symmetry on $AdS_5 \times S^5$ and then show that in AdS_2 it is not reducible.

In $AdS_5 \times S^5$ the fermionic coordinates $\theta^\alpha, \theta^{\hat\alpha}$ transform under κ-symmetry as follows [1]. Working out the algebra in the transformations (2.17), we find

$$\delta\theta^\alpha = (A_z)^\alpha_{\widehat\beta}(\kappa^z)^{\widehat\beta}, \qquad \delta\theta^{\hat\alpha} = (A_{\bar z})^{\hat\alpha}_\beta(\kappa^{\bar z})^\beta, \tag{6.3}$$

where we picked the worldsheet conformal gauge. Here, $\kappa^{\widehat\beta}$ and κ^β are the fermionic gauge parameters; each one of them has sixteen real components of a MW spinor and a holomorphic or anti-holomorphic vector index. The crucial point is the presence of the (field dependent) matrices $A_z, A_{\bar z}$, whose explicit form is $(A_z)^\alpha_{\widehat\beta} = J^m_z(\gamma_m)_{\hat\alpha\widehat\beta}P^{\alpha\widehat\beta}$ and $(A_{\bar z})^{\hat\alpha}_\beta = -J^m_{\bar z}(\gamma_m)_{\alpha\beta}P^{\beta\hat\alpha}$, where $P^{\alpha\hat\alpha} = \frac{1}{2}\delta^{\alpha\hat\alpha}$ is the RR five-form flux. Due to the Virasoro constraints $J^m_z J_{zm} = 0 = J^m_{\bar z}J_{\bar z m}$, these two matrices are not invertible, and in fact they have rank eight, rather than sixteen (it is easy to see this, e.g. because they are nilpotent). As a result, only eight degrees of freedom can be removed by this gauge symmetry. The following choice of the gauge parameters

$$\kappa^z_{(3)} = -[J_{2z}, \epsilon^z_{(1)}], \qquad \kappa^{\bar z}_{(1)} = [J_{2\bar z}, \epsilon^{\bar z}_{(3)}], \tag{6.4}$$

in fact, gives $\delta\theta^\alpha = \delta\theta^{\hat\alpha} = 0$. Similar considerations apply to the cases $d = 4, 6$.

In two dimensions, the spacetime fermionic coordinates are MW spinors with one real component. Therefore, the (non-invertible) matrices $(A_z)^\alpha_{\widehat\beta}$ and $(A_{\bar z})^{\hat\alpha}_\beta$ of (6.3) are replaced by ordinary functions. If we consider the case of the supercoset $Osp(2|2)/\,\mathrm{SO}(2)^2$, the transformations (2.17) read[18]

$$\delta\theta^\alpha = -E^+_z\kappa^{z\alpha}, \qquad \delta\bar\theta^{\hat\alpha} = -E^-_{\bar z}\bar\kappa^{\bar z\hat\alpha}. \tag{6.5}$$

[18] The indices $(\alpha, \hat\alpha)$ are not spinor indices, but simply label the multiplicity of the fermionic coordinates with the same chirality. The case $Osp(1|2)/\,\mathrm{SO}(2)$ considered in [22] is recovered by simply dropping the indices α, $\hat\alpha$.

The special choice (6.4) still gives $\delta\theta^\alpha = \delta\bar{\theta}^{\hat{\alpha}} = 0$. However, now each coordinate can be gauged away independently and the gauge parameter is not multiplied by a nilpotent matrix, but rather by just an ordinary function. Hence, κ-symmetry in two dimensions is not reducible and gauges away all the fermionic coordinates. The counting of classical degrees of freedom thus agrees with target space supersymmetry, leaving an empty classical spectrum.

This can be understood at the level of the constraint algebra as well. Consider first the ten-dimensional GS action S_{GS}. Since the conjugate momenta p's to the θ's do not involve time derivative, we find the so called GS constraint

$$d_\alpha = p_\alpha - \frac{\delta S_{GS}}{\delta\dot{\theta}^\alpha} \approx 0, \qquad \bar{d}_{\hat{\alpha}} = \bar{p}_{\hat{\alpha}} - \frac{\delta S_{GS}}{\delta\dot{\bar{\theta}}^{\hat{\alpha}}} \approx 0,$$

that satisfy the classical algebra

$$\{d_\alpha, d_\beta\} = -\Pi^m(\gamma_m)_{\alpha\beta}, \qquad \{\bar{d}_{\hat{\alpha}}, \bar{d}_{\hat{\beta}}\} = -\bar{\Pi}^m(\gamma_m)_{\hat{\alpha}\hat{\beta}}. \tag{6.6}$$

Each GS constraint is a ten-dimensional MW spinor with sixteen real components. Eight of these components are first class constraints that generate the κ-symmetry transformations in (6.3). The remaining eight components are second class constraints. However, the first and second class constraints are mixed and it is not possible to disentangle them in a manifestly covariant way. This can be seen in the algebra (6.6), since the right hand sides are not vanishing but they are nilpotent matrices on the Virasoro constraints $\Pi^m\Pi_m = 0 = \bar{\Pi}^m\bar{\Pi}_m$. In other words, the right hand sides of the constraint algebra are projectors.

In two dimensions the situation is simpler. We still have two sets of constraints (6.6), whose algebra now reads

$$\{d_\alpha, d_\beta\} = -\delta_{\alpha\beta}E_z^+, \qquad \{\bar{d}_{\hat{\alpha}}, \bar{d}_{\hat{\beta}}\} = -\delta_{\hat{\alpha}\hat{\beta}}\bar{E}_{\bar{z}}^-. \tag{6.7}$$

The claim is that all the GS constraints are now first class. In fact, in two dimensions the Virasoro constraints are $E_z^+E_z^- = 0 = \bar{E}_{\bar{z}}^+\bar{E}_{\bar{z}}^-$, because we just have the two light-cone directions. Therefore a consistent solution to the Virasoro constraint is $E_z^+ = 0 = \bar{E}_{\bar{z}}^-$. As a result, the algebra of the fermionic constraints now closes on first class constraints, namely it is weakly zero. So there are no second class constraints. Now all the d's are generators of the κ-symmetry (6.5), by which we can remove all the fermionic variables.

Pure spinor sigma-model. The action of the pure spinor sigma-model is given by (3.19), where the pure spinor $\beta\gamma$-system is defined according to (3.12). The left and right moving pure spinors λ^α and $\bar{\lambda}^{\hat{\alpha}}$ satisfy the pure spinor constraints (3.2)

$$\lambda^\alpha\delta_{\alpha\beta}\lambda^\beta = 0, \qquad \bar{\lambda}^{\hat{\alpha}}\delta_{\hat{\alpha}\hat{\beta}}\bar{\lambda}^{\hat{\beta}} = 0. \tag{6.8}$$

This is the pure spinor space for two-dimensional type II non-critical superstrings that was introduced in [14, 13, 11], to which we refer for more details (the different notations are explained in the footnote 11).

Holography and isometries. The type IIA superstring on AdS_2 has a natural candidate for the gauge theory dual, living on the boundary of AdS_2. It is a superconformal matrix quantum mechanics with global symmetry group $Osp(2|2)$, corresponding to the global symmetries of the worldsheet theory. The gauge theory is the worldvolume theory living on a stack of many D-particles of the type IIA superstring and is described by a Marinari-Parisi quantum mechanics.

A new feature of holography in our setup is that not all the global symmetries of the dual gauge theory come from isometries of the closed string background. In the usual example of AdS/CFT, the duality is between type IIB superstring theory on $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ SYM theory in four dimensions. The SU(4) R-symmetry of the four-dimensional gauge theory corresponds on the closed string side to the isometry group $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$ of the compact manifold S^5. Consider now the AdS_2 non-critical superstring. It is invariant with respect to a global SO(2) symmetry, generated by $\widetilde{\mathbf{H}}$.[19] This symmetry corresponds again to the R-symmetry on the gauge theory side. However, this rotation does not correspond to any isometry of the closed string background, still it is a global symmetry of the closed string theory. The interpretation of this kind of non-geometric symmetry would fit with the intuition coming from the holography in the case of gauged supergravity. In the gravity spectrum, in that case, there are some additional gauge fields which couple to dual gauge theory operators and explain the extra gauge symmetry. On the string side there are some vertex operators with the correct R-charge assignment, that we could scatter to reproduce the gauge theory computation.

The $Osp(1|2)$ supercoset. We would like to make some comments here on a different realization of type IIA non-critical superstrings on AdS_2 background that was proposed by Verlinde [22]. The action is based on the supercoset $Osp(1|2)/\,\mathrm{SO}(1,1)$. The algebra of $Osp(1|2)$ can be easily obtained from the one of $Osp(2|2)$ (that we list in the appendix) by dropping the indices $\alpha, \hat{\alpha}$, thus removing half of the fermionic generators and discarding the bosonic generator $\widetilde{\mathbf{H}}$. The sigma-model constructed in [22] can be recast in the usual Green-Schwarz like form (2.10) using the grading zero Maurer-Cartan identity relating the exterior product of the bosonic Cartan one-forms with that of the fermionic one-forms. Then, one can apply the machinery developed in this paper to prove that the Green-Schwarz sigma-model still has an infinite number of nonlocal conserved charges, precisely of the form given in (2.29). As a result, the classical Green-Schwarz superstring on the supercoset $Osp(1|2)$ is well defined and integrable.

The pure spinor sigma-model is still given formally by the action (3.19). But when we try to identify the pure spinor variables, we encounter the following feature. The left- and right-moving pure spinors λ and $\bar{\lambda}$ are Weyl spinors of opposite chirality which satisfy the pure spinor constraint

$$\lambda^2 = 0, \qquad \bar{\lambda}^2 = 0. \tag{6.9}$$

[19] Even if we eventually quotient by this generator, the global symmetry of the superstring is the full $Osp(2|2)$. In the same way, in the $AdS_5 \times S^5$ example, the superstring is invariant with respect to the full supergroup $PSU(2,2|4)$.

This fits in the general discussion of section 3.1, by noting that the supersymmetry algebra is now generated by the superderivatives d and $\bar{d}$ satisfying

$$\{d, d\} = -E_z^+, \qquad \{\bar{d}, \bar{d}\} = -\bar{E}_{\bar{z}}^- . \tag{6.10}$$

The solution of the $Osp(1|2)$ pure spinor constraint (6.9) requires that $\lambda = \psi_1\psi_2$ using the two Grassmann odd fields ψ_1, ψ_2.

In the pure spinor sigma-model, the pure spinor variables are interpreted as the ghosts. Consider for simplicity the left sector only (the closed string is the product of the left and right sectors). Then, the physical cohomology is described by operators of ghost number one and weight zero, which in this case are $\mathcal{U} = \lambda A(\theta, x^\pm)$, for a generic superfield A depending on the zero modes only. On general grounds, the prescription for the tree level amplitudes in the pure spinor formalism requires the insertion of three unintegrated vertex operators of ghost number one

$$\mathcal{A} = \langle \mathcal{U}^{(1)}\mathcal{U}^{(2)}\mathcal{U}^{(3)} \ldots \rangle_{CFT}, \tag{6.11}$$

where the dots stand for a generic product of integrated vertex operators.[20] However, all of these tree-level amplitudes include products of three pure spinors which vanish due to the pure spinor constraint.

6.2 Non-critical AdS_4

The non-critical type IIA superstring on AdS_4 with RR four-form flux is realized as a sigma-model on the $Osp(2|4)/\,\mathrm{SO}(1,3) \times \mathrm{SO}(2)$ supercoset. The $Osp(2|4)$ superalgebra and structure constants are discussed in the appendix. The bosonic generators are the translations $\mathbf{P}_a$, the $\mathrm{SO}(1,3)$ generators $\mathbf{J}_{ab}$, for $a, b = 1, \ldots, 4$ and the $\mathrm{SO}(2)$ generator $\mathbf{H}$. The fermionic generators are the supercharges $\mathbf{Q}_\alpha, \mathbf{Q}_{\hat{\alpha}}$, where $\alpha, \hat{\alpha} = 1, \ldots, 4$ are four-dimensional Majorana spinor indices. We have thus $\mathcal{N} = 2$ supersymmetry in four dimensions. The charge assignment of the generators with respect to the $\mathbb{Z}_4$ automorphism of $Osp(2|4)$ can be read from the Maurer-Cartan one forms

$$J_0 = J^{ab}\mathbf{J}_{ab} + J^H\mathbf{H}, \quad J_1 = J^\alpha \mathbf{Q}_\alpha, \quad J_2 = J^a\mathbf{P}_a, \quad J_3 = J^{\hat{\alpha}}\mathbf{Q}_{\hat{\alpha}}. \tag{6.12}$$

The non-critical Green-Schwarz sigma-model on AdS_4 was first introduced in [3, 11]. Again, it is given by (2.10), with the appropriate definitions of the supertrace

$$\langle \mathbf{P}_a\mathbf{P}_b \rangle = \eta_{ab}, \qquad \langle \mathbf{Q}_\alpha \mathbf{Q}_{\hat{\alpha}} \rangle = 2\widetilde{C}_{\alpha\hat{\alpha}}, \tag{6.13}$$

where $\widetilde{C}_{\alpha\hat{\alpha}}$ is an antisymmetric matrix numerically given by the four-dimensional charge conjugation matrix.

The pure spinor sigma-model, which was first introduced in [11], is given by (3.19), where the pure spinor $\beta\gamma$-system is defined according to (3.12). The left and right moving pure spinors λ^α and $\bar{\lambda}^{\hat{\alpha}}$ are four-dimensional Dirac spinors, satisfying the pure spinor

[20] The generalization of the ten-dimensional saturation rule to the non-critical superstring was briefly discussed in [11].

constraints (3.5). This is the pure spinor space for four-dimensional type II non-critical superstrings that was discussed in [11], to which we refer for further details. In the case in which the RR flux is space-filling, there is a subtlety in the definition of the action, in particular in the coupling $\langle d\bar{d}\rangle$. In ten dimensions [39], this part of the action couples the RR superfield $P^{\beta\widehat{\beta}}$ to the fermionic variables d_β and $\bar{d}_{\widehat{\beta}}$ as simply $d_\beta P^{\beta\widehat{\beta}}\bar{d}_{\widehat{\beta}}$. In the AdS_4 case, the relation between the RR superfield and the four-form field strength is

$$P^{\alpha\widehat{\beta}} = \frac{1}{4!} g_S (\widetilde{C}\gamma_{m_1\dots m_4})^{\beta\widehat{\beta}} F_{m_1\dots m_4} = g_S N_c (\widetilde{C}\gamma^5)^{\beta\widehat{\beta}}. \tag{6.14}$$

Since the RR bispinor is proportional to γ^5, it is a pseudoscalar quantity. On the other hand, we want the worldsheet action to be a spacetime scalar, therefore in the GS action (2.10) the correct coupling is $d_\alpha(\gamma^5 P)^{\alpha\hat{\alpha}}\bar{d}_{\hat{\alpha}}$. Since $(\gamma_5)^2 = -\mathbb{I}$, we can again relate the sigma-model on the supergroup with the background fields as explained in the appendix. Notice that the γ_{D+1} is present in the $\langle d\bar{d}\rangle$ part of the action whenever the RR flux is space-filling and the spacetime dimension is even, because in this case the RR bispinor superfield is proportional to the product of all the gamma matrices. In the two-dimensional case, however, we did not underline this subtlety, because we used a one-dimensional spinor notation.

The theory dual to this closed superstring is a strongly coupled three-dimensional SCFT with $\mathcal{N} = 2$ supersymmetry and U(1) R-symmetry. Note that the R-symmetry is realized in a non-geometric way on the string side. It would be interesting to identify the dual to this string theory and to study how the holographic map works in the non-critical string.

6.3 Non-critical $AdS_5 \times S^1$ with open strings

The $AdS_5 \times S^1$ background with five-form flux can be realized as the supercoset

$$AdS_5 \times S^1 = \frac{\mathrm{SU}(2,2|2)}{\mathrm{SO}(1,4)\times\mathrm{SO}(3)}. \tag{6.15}$$

The type IIB superstring theory on this background is not expected to be consistent. Even if it is one-loop conformally invariant, the beta-function for the Weyl invariance should be nonzero [19]. In this section, we will first describe the closed superstring sigma-model on the supercoset (6.15). In appendix D we will speculate about a possible realization of this background as a strongly coupled fixed point without the need of adding open strings.

The bosonic subgroup of SU(2,2|2) is SO(2,4) × SO(3) × U(1). It has nineteen bosonic generators: the five translations $\mathbf{P}_a$ along AdS_5 and the translation $\mathbf{R}$ along the circle S^1, the ten angular momenta $\mathbf{J}_{ab}$ in AdS_5 and the three bosonic generators $\mathbf{T}'_a$ of SO(3). The sixteen fermionic generators are given by the two supercharges $\mathbf{Q}_{\alpha\alpha'}, \mathbf{Q}_{\hat{\alpha}\hat{\alpha}'}$, where the unprimed indices $\alpha, \hat{\alpha} = 1,\dots,4$ are five-dimensional spinor indices in the Majorana representation and the primed indices $\alpha', \hat{\alpha}' = 1,2$ are SO(3) spinor indices. The superalgebra and its structure constants are listed in appendix C. The grading assignment of the generators can be read off the following Maurer-Cartan forms

$$\begin{aligned} J_0 &= J^{ab}\mathbf{J}_{ab} + J^{a'}\mathbf{T}_{a'}, \quad & J_2 &= J^a\mathbf{P}_a + J^R\mathbf{R}, \\ J_1 &= J^{\alpha\alpha'}\mathbf{Q}_{\alpha\alpha'}, & J_3 &= J^{\hat{\alpha}\hat{\alpha}'}\mathbf{Q}_{\hat{\alpha}\hat{\alpha}'}. \end{aligned} \tag{6.16}$$

The κ-symmetric Green-Schwarz sigma-model is again constructed as in (2.10), with the appropriate definitions of the supertrace

$$\langle \mathbf{P}_a \mathbf{P}_b \rangle = -\eta_{ab}, \quad \langle \mathbf{R}\mathbf{R} \rangle = 1, \tag{6.17}$$

$$\langle \mathbf{Q}_{\alpha\alpha'} \mathbf{Q}_{\hat{\alpha}\hat{\alpha}'} \rangle = \widetilde{C}_{\alpha\hat{\alpha}} \widetilde{C}_{\alpha'\hat{\alpha}'}, \tag{6.18}$$

where $\widetilde{C}_{\alpha\hat{\alpha}}$ and $\widetilde{C}_{\alpha'\hat{\alpha}'}$ are antisymmetric matrices numerically given by the charge conjugation matrices of SO(1,4) and of SO(3). This Green-Schwarz action, although in a slightly different form, was discussed in [3, 5]. The classical sigma-model action is κ-symmetric and its one-loop conformal beta-function vanishes.

However, by looking at the non-critical supergravity equations of motion [19], we know that we have to add an open string sector, namely space-filling D-branes, in order to properly cancel the Weyl anomaly, which from the spacetime point of view is encoded in the dilaton equation of motion. We review in appendix D the target space computation of [19]. The computation of the Weyl anomaly, which fixes the radius of the AdS, is tantamount to the evaluation of the central charge of the quantum sigma-model at the strongly coupled fixed point. As in the other examples, we would need strong coupling techniques to address this question, which are lacking at the moment.

In order to study the quantum sigma-model, we can introduce the pure spinor formulation of the supercoset by considering the action (3.19), where the pure spinor $\beta\gamma$-system is defined according to (3.12). The left and right moving pure spinors $\lambda^{\alpha\alpha'}$ and $\bar{\lambda}^{\hat{\alpha}\hat{\alpha}'}$ are five-dimensional symplectic Majorana spinors (with a corresponding SO(1,4) spinor index α or α') and have an extra index α' and $\hat{\alpha}'$ in the spinor representation of SO(3).[21] The six-dimensional pure spinor constraint (3.9), rewritten in terms of the supercoset (6.15), reads

$$\begin{aligned} C'_{\alpha'\beta'} \lambda^{\alpha\alpha'} (C\gamma^m)_{\alpha\beta} \lambda^{\beta\beta'} &= 0, \qquad m = 0, \ldots, 4, \\ C'_{\alpha'\beta'} C_{\alpha\beta} \lambda^{\alpha\alpha'} \lambda^{\beta\beta'} &= 0, \end{aligned} \tag{6.19}$$

where γ^m and C are the SO(1,4) gamma matrices and charge conjugation matrix and C' is the SO(3) charge conjugation matrix. The pure spinor action is again BRST invariant. As for the κ-symmetry discussed before, it seems that in lower-dimensional pure spinor superstrings, the BRST symmetry is related to the one-loop conformal invariance but not to the Weyl invariance.

Adding open strings. The pure spinor sigma-model we have constructed above, even if it is gauge invariant and BRST invariant at all order in perturbation theory, does not correspond to a consistent non-critical superstring. It is only consistent after adding an open string sector. In particular, [19] suggested introducing boundary conditions corresponding to uncharged space-filling D-branes. We need uncharged D-branes because we

[21] The original six-dimensional pure spinors $\lambda^{\alpha i}$ of (3.9) is in the Weyl representation of SO(6) and has an additional index $i = 1, 2$ transforming as a doublet of SU(2). It decomposes naturally according to the local symmetry group of our supercoset. The Weyl representation of SO(6) corresponds to the symplectic Majorana representation of $SO(1,4) \simeq Sp(4)$, while the extra harmonic index i corresponds precisely to the spinor index α' of SO(3).

do not want to introduce any additional RR flux. They can be thought of as space-filling brane-antibrane pairs.

When we put simultaneously D-brane and anti-D-brane boundary conditions in a flat background, two things usually happen. This completely breaks spacetime supersymmetry, since they preserve two different sets of supercharges. An open string tachyon appears in the spectrum. However, it was argued that, in the case of space-filling branes and anti-branes on $AdS_5 \times S^1$, the physics is different from the flat space one. In particular, the space-filling brane anti-brane system will break only half of the sixteen spacetime supersymmetries.[22] We suggest that this system will break the global SU(2) symmetry of the supercoset as well, leaving just a bosonic $\mathrm{SO}(2,4) \times \mathrm{U}(1)$ global symmetry. Moreover, the mass squared of the open string tachyon, albeit negative, will be above the BF bound and therefore lead to no instabilities. It would be interesting to prove these two conjectures, by studying the spectrum of the worldsheet theory we have just described. We leave this for future investigations.

It was suggested in [19] that the gauge theory dual to this closed plus open superstring theory be four-dimensional $\mathcal{N} = 1$ SQCD at an IR superconformal fixed point. Note that holography usually relates a closed superstring theory to a gauge theory, while here we are considering a closed plus open superstring theory. The N_f brane anti-brane pairs correspond to the gauge theory flavors. Provided that the two conjectures we discussed above are indeed verified, the global symmetries on the two sides of the duality are matched. The string theory global symmetries are $\mathrm{SO}(2,4) \times \mathrm{U}(1)_R$, coming from the $AdS_5 \times S^1$ isometries, and an additional $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ flavor symmetry group that rotates the space-filling branes and anti-branes. This fits nicely with the global symmetry group of SQCD at the IR fixed point.

The first step in establishing this holographic duality would be to compute the mass of the open string tachyon, which will depend on the RR five-form flux N_c and the number of flavor branes N_f. One expects that it satisfies the BF bound when N_f and N_c are inside the conformal window of the dual gauge theory.

6.4 Ten-dimensional $AdS_p \times S^p \times \mathcal{M}$

All the backgrounds of the kind $AdS_p \times S^p$ with RR p-form flux correspond to the noncompact part of a ten-dimensional background.[23] This is because the scalar curvature of these

[22] In the six-dimensional linear dilaton background, the system of N_f space-filling brane/antibrane pairs, together with N_c D3 branes extending in the flat Minkowski part of the space was studied in [40] and later inn [41]. For a finite number of colors and flavors, the system preserves four supersymmetries, even if branes and anti-branes are present simultaneously. We can regard the $AdS_5 \times S^1$ background as the near horizon limit of the D3 branes of the linear dilaton system, when the number of colors and flavors becomes very large. As in the usual $AdS_5 \times S^5$ case, in the near horizon limit we double the number of supersymmetries, which gives a total of eight supercharges, the appropriate number to match the dual four-dimensional $\mathcal{N} = 1$ SQCD in the conformal window.

[23] They have been studied in the hybrid formalism in [29] for $p = 2$ and in [33] for $p = 3$. The matter part of the hybrid action is the same as the matter part of the pure spinor action. However their ghost sector is different.

spaces vanishes so they satisfy the one-loop beta-function equations with $D = 10$.[24] The case $p = 5$ corresponds to the well known $AdS_5 \times S^5$ background [1]. The lower-dimensional cases $p = 2, 3$, are suitable for a Calabi-Yau compactification on a three- and a two-fold, respectively.

Curvature equation. In [29] it was shown that if G is a Ricci flat supergroup, then the scalar curvature of the supercoset G/H is equal to the curvature of its bosonic subgroup, namely the bosonic $AdS_p \times S^p$ manifold. The AdS and the sphere give the same contribution but with opposite sign,[25] so the total scalar curvature of the supercoset vanishes. We will review now how this cancellation works at the level of the super Ricci curvature itself.

We denote by $a, b = 1, \ldots, p$ the vector indices along AdS_p and by $a', b' = 1, \ldots, p$ the vector indices along the S^p. The bosonic legs of the super Ricci curvature are

$$\begin{aligned} R_{ab} &= \frac{1}{16}\{\gamma_a, \gamma_b\}^\alpha{}_\alpha \delta^{\alpha'}{}_{\alpha'} - \eta_{as}\delta^c{}_c + \eta_{ab} \\ &= \left(\frac{1}{8} S_{AdS_p} S_{S^p} - V_{AdS_p} + 1\right)\eta_{ab}, \end{aligned} \tag{6.20}$$

$$R_{a'b'} = -\frac{1}{16}\{\gamma_{a'}, \gamma_{b'}\}^{\alpha'}{}_{\alpha'} \delta^\alpha{}_\alpha - \eta_{a'b'}\delta^{c'}{}_{c'} + \eta_{a'b'} \tag{6.21}$$

$$= -\left(\frac{1}{8} S_{S^p} S_{AdS_p} - V_{S_p} + 1\right)\eta_{a'b'} \tag{6.22}$$

where η_{ab} and $\eta_{a'b'}$ are the bosonic metrics on the supergroup and α are the spinor indices on AdS_p while α' are the spinor indices on S^p. S_{AdS_p} and S_{S_p} stand for the dimension of the spinor representation of the AdS_p and S^p part of the super-algebra, similarly V_{AdS_p} and V_{S_p} are the bosonic dimensions of the AdS_p and S^p spaces. It is clear that the bosonic AdS_p and the S^p contributions to the scalar curvature cancel each other. For the Fermionic part of the super Ricci curvature we have

$$\begin{aligned} R(G/H)_{\alpha\alpha'\widehat{\beta}\widehat{\beta}'} &= \frac{1}{8}\Big[- C'_{\alpha'\widehat{\beta}'}(C\gamma^a\gamma_a)_{\alpha\widehat{\beta}} + C_{\alpha'\widehat{\beta}'}(C'\gamma^{a'}\gamma_{a'})_{\alpha\widehat{\beta}} - C'_{\widehat{\beta}'\alpha'}(C\gamma^a\gamma_a)_{\widehat{\beta}\alpha} + \\ &\quad + C_{\widehat{\beta}'\alpha'}(C'\gamma^{a'}\gamma_{a'})_{\widehat{\beta}\alpha}\Big] - \frac{1}{2}\left(C'_{\alpha'\widehat{\beta}'}(C\gamma^{cd}\gamma_{cd})_{\alpha\widehat{\beta}} + C_{\alpha'\widehat{\beta}'}(C'\gamma^{c'd'}\gamma_{c'd'})_{\alpha\widehat{\beta}}\right) = \\ &= \frac{1}{2}\Big[(V_{S^p})^2 - (V_{AdS_p})^2\Big] C'_{\alpha'\widehat{\beta}'} C_{\alpha\widehat{\beta}}, \end{aligned} \tag{6.23}$$

where C and C' are the charge conjugation matrices of the AdS_p and S^p part of the supergroup. We find that the fermionic part of the super Ricci curvature vanishes identically

$$R(G/H)_{\alpha\alpha'\widehat{\beta}\widehat{\beta}'} = 0. \tag{6.24}$$

As a result, the supertrace of the super Ricci curvature vanishes.

[24]This agrees with the results of [42]. Those authors found that there is no solution to the leading order non-critical supergravity equations for these backgrounds when supported only by RR flux.

[25]Note that in the supercoset construction the radii of AdS_p and S^p need to be equal in order to preserve the superalgebra.

6.5 Central charge

In order to verify that our models are consistent string theories we need to see that their total central charge vanishes. Since the sigma-models are strongly coupled, it is hard to compute the exact central charge at their fixed point. What we can do is to consider the "naive" central charge that one would get in the small curvature limit, i.e., in the classical sigma-model. In the pure spinor sigma-model, we just add up the contribution to the central charge coming from each CFT separately, since in the small curvature limit they are just free and decoupled. We are in fact just computing the flat space central charge. In each sector, the matter part is given by the bosons $\{X^a\}$, for $a = 1, \ldots, 2d$, which are worldsheet scalars, and the supercoordinates and their conjugate momenta $\{p_\alpha, \theta^\alpha\}$, which have weight $(1,0)$, while the pure spinor beta-gamma system $\{w_\alpha, \lambda^\alpha\}$ has weight $(1,0)$ and has been described in section 3.1. It turns out that in all different dimensions, the matter central charge is exactly cancelled by the ghost central charge. The field content is the same for both the non-critical and the critical models (see [11] and [14, 13]). We summarize the various contribution to the vanishing central charge in different dimensions as follows

$$\begin{array}{llllll} d=2: & c_{\text{tot}} = (2)_{\{X\}}+ & (-4)_{\{p,\theta\}} & +(2)_{\{w,\lambda\}} & = 0, \\ d=4: & c_{\text{tot}} = (4)_{\{X\}}+ & (-8)_{\{p,\theta\}} & +(4)_{\{w,\lambda\}} & = 0, \\ d=6: & c_{\text{tot}} = (6)_{\{X\}}+ & (-16)_{\{p,\theta\}} & +(10)_{\{w,\lambda\}} & = 0. \end{array} \tag{6.25}$$

This counting of degrees of freedom is of course heuristic. The evaluation of the exact central charge requires to solve for the spectrum of the model. One way of doing it is by making use of the Bethe ansatz approach developed in [43]. We hope to report about this in the future.

7. Discussion and open problems

In this paper we have constructed the worldsheet theory of type II superstrings on AdS backgrounds with RR flux, which are realized as supercosets G/H where G has a $\mathbb{Z}_4$ automorphism. We have shown in particular that in all such backgrounds string theory is quantum integrable. This holds both for the non-critical and for the ten-dimensional superstrings, in particular for the topological sector of the latter. A nice feature we found[26] is that the dependence of the Lax connection $a(\mu)$ on the spectral parameter (3.35) is the same in all models. Once we established the existence of these type II backgrounds, there are many directions that open up for a future investigation. Let us list some of them.

The worldsheet theory for non-critical strings is a strongly coupled sigma-model, whose coupling is given by the curvature of AdS. Due to the lack of tools to analyze strongly coupled sigma-models we could prove neither exact conformal invariance nor Weyl invariance. It would be interesting to prove the existence of the fixed point non-perturbatively, or at least to all orders in perturbation theory.[27] In the case of AdS_2 more can be said. The

[26]This was also noticed in [5] for the GS sigma models.

[27]For the backgrounds of the kind $AdS_p \times S^p$, the beta-function vanishes at all orders in perturbation theory [33]. However, the method used in those cases relies on an extension of the supergroup G which is not possible in our non-critical cases.

proofs of gauge invariance, BRST invariance and integrability of the sigma-model that we discussed above hold in general to all orders in the worldsheet perturbation theory. The non-perturbative contributions to the action, on the other hand, come in the form of worldsheet instantons and are counted by the factor $e^{-\frac{1}{\lambda^2}}$, where λ is the sigma-model coupling. In the AdS_2 case there are no two-cycles the worldsheet instantons can wrap on. Hence, the sigma-model does not receive any non-perturbative corrections and it is gauge invariant, BRST invariant and integrable exactly.

Once it is established that these superstring sigma-models are integrable, it is natural to look for their spectrum. The computation of the quantum spectrum of string theory on $AdS_5 \times S^5$ based on the $PSU(2,2|4)$ supercoset is a formidably hard task. Our lower dimensional non-critical string theories might be easier to solve since they are described by somewhat simpler supercosets. The AdS_2 background is probably the simplest example of a type II RR background and we have argued that the sigma-model is exact non-perturbatively, due to the absence of worldsheet instantons. Since the spacetime is two-dimensional, the semiclassical spectrum is empty. The next non-trivial example is the AdS_4 background, for which the semiclassical spectrum contains for example spinning string solutions. As a first step, it would be interesting to work out the complete classical spectrum, encoded in the algebraic curve method of [16], which fully exploits the integrability properties of the classical sigma-model. In order to look for the spectrum of the quantum sigma-model, one can follow two different approaches. A first way is by computing the pure spinor cohomology. The second approach makes use of the Bethe ansatz, as proposed in [43]. In the latter paper, the authors focussed on a toy model, based on the supercoset $Osp(2m+2|2m)$. It seems plausible that our sigma-model for AdS_4, which we realized as a $Osp(2|4)$ supercoset, might be solvable in the same spirit. The exact solution will fix also the value of the central charge at the strongly coupled fixed point, which we have not been able to evaluate.

Another issue pertains to the interpretation of the ten-dimensional backgrounds $AdS_p \times S^p \times CY_{5-p}$, whose non-compact part we have discussed in detail. While their GS formulation certainly describes the full compactified superstring, the interpretation of the pure spinor formulation is still not completely clear. Recently, there have been different proposals regarding the pure spinor superstring compactified on Calabi-Yau [24, 14, 13, 28, 34]. In the case in which the background is flat four-dimensional Minkowski times a CY three-fold, it has been argued that the pure spinor formalism computes only the topological amplitudes of the full superstring [24]. It would be interesting to understand what happens in our backgrounds.

In section 3.4 we considered the addition of boundary conditions to the sigma-model, in particular space-filling D-branes. The classification of branes in non-compact spaces can be in general a hard task, even when they are supported only by NS-NS flux. Since very little is known about such classification on RR backgrounds, it would be nice to make progress in this analysis. In particular, the pure spinor formalism seems a convenient starting point for such a search, due to his simple couplings to RR backgrounds.

An interesting open problem is to figure out how holography works for the non-critical backgrounds and to identify the field theory duals to these new non-critical AdS_{2d} back-

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grounds if they are field theories at all. In particular, the existence of the infinite set of nonlocal charges on the string sigma-model is related to the Yangian symmetry of the dual gauge theory. In the $AdS_5 \times S^5$ case, the existence of the Yangian symmetry has been established in the dual $\mathcal{N} = 4$ gauge theory at weak coupling [17]. It would be nice to identify the Yangian symmetry on the gauge dual side in all our cases.

The duals of the non-critical superstrings on AdS_{2d} are in general strongly coupled $2d-1$ superconformal field theories. A particularly interesting example would be the relation between the type IIB non-critical superstring on $AdS_5 \times S^1$ with space-filling branes and four-dimensional $\mathcal{N} = 1$ SQCD, which was suggested in [19] by looking at the six-dimensional non-critical supergravity. In this case there is no decoupling limit, namely the string dual of the conformal window of SQCD should contain closed as well as open strings. It would be nice to analyze our worldsheet theory for such background. Firstly, one should check the vanishing of the one-loop beta-function in the presence of boundaries. Only with the contribution coming from the boundary of the worldsheet should the theory be Weyl invariant at one-loop. This is equivalent to the statement, reviewed in appendix D, that the non-critical supergravity equations of motion for both the metric and the dilaton are satisfied only with the inclusion of the space-filling branes. Then, one should show that, at least perturbatively in the sigma-model coupling, the mass of the open string tachyon lies above the BF bound. An interesting thing to study is T-duality along the circle, which might be related to Seiberg duality in the dual SQCD [19].

Finally, it would be interesting to see what happens to the Yangian symmetry of the closed string sector once we add boundaries. The Bethe ansatz for open strings on $AdS_5 \times S^5$ was discussed in [44] and in a recent paper [45] it has been shown that for certain choices of boundary conditions the bosonic part of the open string sector is still integrable. It would be interesting to check whether this is true for the $AdS_5 \times S^1$ model with space-filling branes. In the case in which the open string sector is integrable, it would be intriguing to investigate the implications of such a symmetry in the dual SQCD.

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A. κ-symmetry and torsion constraints

In section 2.1 we constructed the Green-Schwarz action on a supergroup with $\mathbb{Z}_4$ automorphism and we found that the structure constants of the supergroup have to obey the relation (2.19) rewritten here

$$\eta_{\beta\hat{\beta}} \left(f^{\beta}_{a\hat{\alpha}} f^{\hat{\beta}}_{b\alpha} + f^{\beta}_{b\hat{\alpha}} f^{\hat{\beta}}_{i\alpha} \right) = c_{\alpha\hat{\alpha}} \eta_{ab}, \tag{A.1}$$

where $c_{\alpha\hat{\alpha}}$ is some symmetric matrix, in order for the action to be κ-symmetric. We will show now how this relation is equivalent to the torsion constraints of the supergravity background.

Recalling that the spacetime torsion is defined as

$$[\nabla_A, \nabla_B\} = T_{AB}^C \nabla_C, \tag{A.2}$$

we find that the structure constants $f_{a\hat{\alpha}}^{\beta}$ and $f_{b\alpha}^{\hat{\beta}}$ correspond to some particular components of the torsion

$$f_{a\alpha}^{\widehat{\beta}} = -T_{a\alpha}^{\widehat{\beta}},\; f_{a\hat{\alpha}}^{\beta} = -T_{a\hat{\alpha}}^{\beta}. \tag{A.3}$$

Let us specialize to the case $AdS_p \times S^p$ first, namely the type II superstring compactified on a Calabi Yau. In type II supergravity the torsion is related to the RR superfield $P^{\beta\widehat{\beta}}$ by the following constraint [46]

$$T_{a\alpha}^{\widehat{\beta}} = (\gamma_a)_{\alpha\beta} P^{\beta\widehat{\beta}},\; T_{a\hat{\alpha}}^{\beta} = (\gamma_a)_{\hat{\alpha}\widehat{\beta}} P^{\beta\widehat{\beta}}, \tag{A.4}$$

and the RR superfield is given by

$$P^{\beta\widehat{\beta}} = \frac{g_S}{p!} (\gamma_{m_1 \ldots m_p})^{\beta\widehat{\beta}} F^{m_1 \ldots m_p} = g_S N_c \delta^{\beta\widehat{\beta}} \tag{A.5}$$

where F_p is the self dual p-form flux and the γ_a are the Pauli matrices, namely the off diagonal blocks of the Dirac matrices of SO$(1, p-1)$. The structure constants are given in appendix C.6As a last ingredient, we notice that the metric on the supergroup is proportional to the inverse of the RR superfield $\eta_{\beta\widehat{\beta}} \propto (P^{-1})_{\beta\widehat{\beta}} \propto \delta_{\beta\widehat{\beta}}$. Putting everything together, we can cast the relation (A.1) in the form

$$\eta_{\beta\hat{\beta}} \left(f_{a\hat{\alpha}}^{\beta} f_{b\alpha}^{\hat{\beta}} + f_{b\hat{\alpha}}^{\beta} f_{i\alpha}^{\hat{\beta}} \right) = \{\gamma_a, \gamma_b\}_{\alpha}^{\beta} \delta_{\beta\hat{\alpha}} = 2\eta_{ab}\delta_{\alpha\hat{\alpha}}, \tag{A.6}$$

so we find that the symmetric matrix $c_{\alpha\hat{\alpha}} = \delta_{\alpha\hat{\alpha}}$ is proportional to the inverse RR flux.

In the case of the non-critical AdS_2, AdS_4 and $AdS_5 \times S^1$, the same result follows, provided an analogous torsion constraint (A.4) is imposed. This can be understood again as a supergravity constraint of $\mathcal{N} = (2, 2)$ in two dimensions [47] or $\mathcal{N} = 2$ in four and six dimensions [46].

B. Pure spinor sigma-models

In this appendix we collect some computations used in the main text for the pure spinor superstrings. We refer to sections 2 and 3 for the notations.

B.1 The pure spinor sigma-model from BRST symmetry

Using the fact that $\langle AB \rangle \neq 0$ only for $A \in \mathcal{H}_r$ and $B \in \mathcal{H}_{4-r}$, $r = 0, \ldots, 3$ [29], the most general matter part which has a global symmetry under left multiplication by elements of G and is invariant under the gauge symmetry $g \simeq gh$, where $h \in H$, is

$$\int d^2z \langle \alpha J_2 \bar{J}_2 + \beta J_1 \bar{J}_3 + \gamma J_3 \bar{J}_1 + \delta J_3 \bar{d} + \epsilon \bar{J}_1 d - f d\bar{d} \rangle \,,$$

where we used the Lie-algebra valued field d, $\bar{d}$ defined by $d = d_\alpha \eta^{\alpha\hat{\alpha}} T_{\hat{\alpha}}$, $\bar{d} = \bar{d}_{\hat{\alpha}} \eta^{\alpha\hat{\alpha}} T_\alpha$ and f is the RR-flux. While in flat background the d's are composite fields, in curved backgrounds they can be treated as independent fields.

The pure spinor part includes the kinetic terms $\langle w\bar{\partial}\lambda\rangle$ and $\langle \bar{w}\partial\bar{\lambda}\rangle$ for the pure spinor $\beta\gamma$-systems. Since these terms are not gauge invariant, they must be accompanied by terms coupling the pure spinor gauge generators with the matter gauge currents $\langle N\bar{J}_0 + \bar{N}J_0\rangle$ in order to compensate. The backgrounds we are considering also require additional terms which must be gauge invariant under the pure spinor gauge transformation of w and $\bar{w}$ and hence must be expressed in terms of the Lorentz currents and the ghost currents $J_{\rm gh} = \langle w\lambda\rangle$ and $\bar{J}_{\rm gh} = \langle \bar{w}\bar{\lambda}\rangle$ (such terms are given by $S^{\beta\hat{\delta}}_{\alpha\hat{\gamma}}$ in the Type II action in [48]). The additional term required is $\langle N\bar{N}\rangle$.

Therefore the sigma-model is of the form

$$S = \int d^2z \langle \alpha J_2\bar{J}_2 + \beta J_1\bar{J}_3 + \gamma J_3\bar{J}_1 + \delta J_3\bar{d} + \epsilon \bar{J}_1 d - f d\bar{d} + + w\bar{\partial}\lambda + \bar{w}\partial\bar{\lambda} + N\bar{J}_0 + \bar{N}J_0 + aN\bar{N}\rangle \tag{B.1}$$

and the accompanying BRST-like operator is

$$Q_B = \oint \langle dz\lambda d - d\bar{z}\bar{\lambda}\bar{d}\rangle \ . \tag{B.2}$$

By integrating out d and $\bar{d}$ and redefining $\gamma \to \gamma + \frac{\epsilon\delta}{f}$ one gets

$$S = \int d^2z \langle \alpha J_2\bar{J}_2 + \beta J_1\bar{J}_3 + \gamma J_3\bar{J}_1 + w\bar{\partial}\lambda + \bar{w}\partial\bar{\lambda} + N\bar{J}_0 + \bar{N}J_0 + aN\bar{N}\rangle \ . \tag{B.3}$$

After rescaling $\lambda \to \frac{\delta}{f}\lambda$, $w \to \frac{f}{c}\delta w$, $\bar{\lambda} \to \frac{\epsilon}{f}\bar{\lambda}$, $\bar{w} \to \frac{f}{\epsilon}\bar{w}$ the BRST currents are $j_B = \langle \lambda d\rangle = \langle \lambda J_3\rangle$ and $\bar{j}_B = \langle \bar{\lambda}\bar{d}\rangle = \langle \bar{\lambda}\bar{J}_1\rangle$. The BRST charge (B.2) now reads

$$Q_B = \oint \langle dz\lambda J_3 + d\bar{z}\bar{\lambda}\bar{J}_1\rangle \ . \tag{B.4}$$

The coefficients of the various terms will be determined by requiring the action to be BRST invariant, i.e. the BRST currents are holomorphic and the corresponding charge is nilpotent.

From the action (B.3) we derive the following equations of motion

$$(\beta+\gamma)\bar{\nabla}J_3 = (2\beta-\alpha)[J_1,\bar{J}_2] + (\alpha+\beta-\gamma)[J_2,\bar{J}_1] + [N,\bar{J}_3] + [\bar{N},J_3] \,, \tag{B.5}$$

$$(\beta+\gamma)\nabla\bar{J}_1 = (\alpha-2\beta)[J_2,\bar{J}_3] + (\gamma-\alpha-\beta)[J_3,\bar{J}_2] + [N,\bar{J}_1] + [\bar{N},J_1] \,, \tag{B.6}$$

$$\bar{\nabla}\lambda = -a[\bar{N},\lambda] \,, \quad \nabla\bar{\lambda} = -a[N,\bar{\lambda}] \ . \tag{B.7}$$

After one takes into account that $[N,\lambda] = 0$ because of the pure spinor condition $\{\lambda,\lambda\} = 0$ [10], requiring $\bar{\partial} j_B = 0$ leads to the equations

$$\beta+\gamma = 1 \,, \quad \alpha = 2\beta \,, \quad \alpha+\beta = \gamma \,, \quad a = -1 \,, \tag{B.8}$$

whose solution is

$$\alpha = \frac{1}{2}\,, \quad \beta = \frac{1}{4}\,, \quad \gamma = \frac{3}{4}\,, a = -1\;. \tag{B.9}$$

With this solution it is easy to check that

$$\partial \bar{j}_B = \langle [\bar{\lambda}, \bar{N}] J_1 \rangle\,, \tag{B.10}$$

which again vanishes because of the constraint $\{\bar{\lambda}, \bar{\lambda}\} = 0$. The proof of the nilpotence of the BRST charge then follows just as in [10].

Hence the pure spinor sigma-model is

$$S = \int d^2z \left\langle \frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 + \frac{3}{4} J_3 \bar{J}_1 + w \bar{\partial} \lambda + \bar{w} \partial \bar{\lambda} + N \bar{J}_0 + \bar{N} J_0 - N \bar{N} \right\rangle \tag{B.11}$$

for all dimensions and this of course matches the critical case as well.

B.2 BRST invariance of the conserved pure spinor charges

We would like to verify that the conserved charges (3.36) are BRST invariant. This requirement stems from the fact that for a charge to be a symmetry, it is not sufficient for it to be conserved. A symmetry maps physical states (states in the pure spinor cohomology) to other physical states. For this to happen, the charge itself must be BRST-closed.

The BRST transformations of the various worldsheet fields are given by

$$\delta_B g = g(\epsilon\lambda + \epsilon\bar{\lambda})\,, \quad \delta_B w = -J_3 \epsilon\,, \quad \delta_B \bar{w} = -\bar{J}_1 \epsilon\,, \quad \delta_B \lambda = \delta_B \bar{\lambda} = 0\,, \tag{B.12}$$

$$\delta_B J_0 = [J_3, \epsilon\lambda] + [J_1, \epsilon\bar{\lambda}]\,, \tag{B.13}$$

$$\delta_B J_1 = \partial(\epsilon\lambda) + [J_0, \epsilon\lambda] + [J_2, \epsilon\bar{\lambda}]\,, \tag{B.14}$$

$$\delta_B J_2 = [J_1, \epsilon\lambda] + [J_3, \epsilon\bar{\lambda}]\,, \tag{B.15}$$

$$\delta_B J_3 = \partial(\epsilon\bar{\lambda}) + [J_2, \epsilon\lambda] + [J_0, \epsilon\bar{\lambda}]\,, \tag{B.16}$$

$$\delta_B N = \{J_3 \epsilon, \lambda\}\,, \quad \delta_B \bar{N} = \{\bar{J}_1 \epsilon, \bar{\lambda}\}\;. \tag{B.17}$$

In order to demonstrate that the charges (3.36) are indeed BRST closed, we define the following operator

$$\mathrm{U}(x, \bar{x}; y, \bar{y}) = \mathrm{P} \exp \left[- \int_y^x (dz a + d\bar{z} \bar{a}) \right]\;. \tag{B.18}$$

The BRST variation of U_C can now be written as

$$\begin{aligned} \delta_B U_C = &- \int_C dz\, \mathrm{U}(x, \bar{x}; z, \bar{z}) \delta_B a(z, \bar{z}) U(z, \bar{z}; y, \bar{y}) - \\ &- \int_C d\bar{z} U(x, \bar{x}; z, \bar{z}) \delta_B \bar{a}(z, \bar{z}) U(z, \bar{z}; y, \bar{y})\,, \end{aligned} \tag{B.19}$$

where

$$\begin{aligned} \delta_B a = g\Big[& c_2([J_1, \epsilon\lambda] + [J_3, \epsilon\bar{\lambda}]) + c_1(\epsilon\partial\lambda + [J_0, \epsilon\lambda] + [J_2, \epsilon\bar{\lambda}]) + \\ & + c_3(\epsilon\partial\bar{\lambda} + [J_2, \epsilon\lambda] + [J_0, \epsilon\bar{\lambda}]) + c_N \{J_3 \epsilon, \lambda\} + [\epsilon\lambda + \epsilon\bar{\lambda}, A]\Big] g^{-1}\,, \end{aligned} \tag{B.20}$$

$$\begin{aligned} \delta_B \bar{a} = g\Big[& \bar{c}_2([\bar{J}_1, \epsilon\lambda] + [\bar{J}_3, \epsilon\bar{\lambda}]) + \bar{c}_1(\epsilon\bar{\partial}\lambda + [\bar{J}_0, \epsilon\lambda] + [\bar{J}_2, \epsilon\bar{\lambda}]) + \\ & + \bar{c}_3(\epsilon\bar{\partial}\bar{\lambda} + [\bar{J}_2, \epsilon\lambda] + [\bar{J}_0, \epsilon\bar{\lambda}]) + \bar{c}_N \{\bar{J}_1 \epsilon, \bar{\lambda}\} + [\epsilon\lambda + \epsilon\bar{\lambda}, \bar{A}]\Big] g^{-1} \end{aligned} \tag{B.21}$$

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and $g : \Sigma \to G$ is the mapping from the worldsheet to the supergroup G.

The derivative terms in the first integral in (B.19) can be computed by integrating by parts, so the derivative terms turn out to be

$$\int_C dz U(x,\bar{x};z,\bar{z}) g(z,\bar{z}) \left[[c_1\epsilon\lambda + c_3\epsilon\bar{\lambda}, A] + [c_1\epsilon\lambda + c_3\epsilon\bar{\lambda}, J]\right]_{z,\bar{z}} g(z,\bar{z})^{-1} U(z,\bar{z};y,\bar{y}) \ .$$

Plugging this back into the integral and collecting all terms one gets

$$\begin{aligned} I = \int_C & dz U(x,\bar{x};z,\bar{z}) g(z,\bar{z}) \Big[(c_2 - c_1(c_1+2))[J_1,\epsilon\lambda] + (c_3 - c_2(c_1+1) - c_1)[J_2,\epsilon\lambda] + \\ & + (c_N - c_3(c_1+1) - c_1)[J_3,\epsilon\lambda] + (c_2 - c_3(c_3+2)[J_3,\epsilon\bar{\lambda}] + c_N(c_1+1)[\epsilon\lambda, N] + \\ & + (c_1 - c_2(c_3+1) - c)[J_2,\epsilon\bar{\lambda}] + (c_1(c_3+1) + c)[\epsilon\bar{\lambda}, J_1] + \\ & + c_N(c_3+1)[\epsilon\bar{\lambda}, N] \Big]_{z,\bar{z}} g(z,\bar{z})^{-1} U(z,\bar{z};y,\bar{y}) \ . \end{aligned} \tag{B.22}$$

Note that $[\lambda, N] = 0$ by using the Jacobi identity and the pure spinor constraint $\{\lambda,\lambda\} = 0$. The last term is handled by using the equation of motion $\nabla\bar{\lambda} = [N, \bar{\lambda}]$

$$\begin{aligned} & \int_C dz U(x,\bar{x};z,\bar{z}) g(z,\bar{z}) \ [\epsilon\bar{\lambda}, N]\big|_{z,\bar{z}} g(z,\bar{z})^{-1} U(z,\bar{z};y,\bar{y}) = \\ & = - \int_C dz U(x,\bar{x};z,\bar{z}) g(z,\bar{z}) \Big[[\epsilon\bar{\lambda}, A] + [\epsilon\bar{\lambda}, J_1 + J_2 + J_3] \Big]_{z,\bar{z}} g(z,\bar{z})^{-1} U(z,\bar{z};y,\bar{y}) \ . \end{aligned}$$

Using this for a fraction x of $[\epsilon\bar{\lambda}, N]$ yields that the integral evaluates to

$$\begin{aligned} I = \int_C & dz U(x,\bar{x};z,\bar{z}) g(z,\bar{z}) \Big[(c_2 - c_1(c_1+2))[J_1,\epsilon\lambda] + (c_3 - c_2(c_1+1) - c_1)[J_2,\epsilon\lambda] + \\ & + (c_N - c_3(c_1+1) - c_1)[J_3,\epsilon\lambda] + (c_2 - c_3(c_3+2) + x(c_3+1))[J_3,\epsilon\bar{\lambda}] + \\ & + (c_1 - c_2(c_3+1) - c_3 + x(c_2+1))[J_2,\epsilon\bar{\lambda}] + (c_1(c_3+1) + c_3 - x(c_1+1))[\epsilon\bar{\lambda}, J_1] + \\ & + (c_N(c_3+1) - x(c_N+1))[\epsilon\bar{\lambda}, N] \Big]_{z,\bar{z}} g(z,\bar{z})^{-1} U(z,\bar{z};y,\bar{y}) \ . \end{aligned} \tag{B.23}$$

After choosing $x = \frac{c_N(c_3+1)}{c_N+1}$ and substituting (3.35) we get $I = 0$. The second integral in (B.19) vanishes in a similar way, so the conserved charges found here are indeed BRST invariant.

B.3 Ghost number one cohomology

In this appendix we will prove the claim made in section 4.2 that the classical BRST cohomology of integrated vertex operators $\int d^2z \langle \mathcal{O}^{(1)}_{z\bar{z}} \rangle$ at ghost number one is empty.

The most general ghost number one gauge-invariant integrated vertex operator is

$$\begin{aligned} \langle \mathcal{O}^{(1)}_{z\bar{z}} \rangle \ = \ & \langle a_1 \bar{J}_2 [J_3, \epsilon\bar{\lambda}] + \bar{a}_1 J_2 [\bar{J}_1, \epsilon\lambda] + a_2 \bar{J}_2 [J_1, \epsilon\lambda] + \bar{a}_2 J_2 [\bar{J}_3, \bar{\lambda}] \\ & + a_3 J_3 [\bar{N}, \epsilon\lambda] + \bar{a}_3 \bar{J}_1 [N, \epsilon\bar{\lambda}] + a_4 J_3 \bar{\nabla}(\epsilon\lambda) + \bar{a}_4 \bar{J}_1 \nabla(\epsilon\bar{\lambda}) \rangle, \end{aligned} \tag{B.24}$$

where we have written all the independent terms up to integrating by parts on the Maurer-Cartan equations. We will consider the insertion of a boundary at the end and concentrate on the bulk terms first. The BRST variation of the operator (B.24) consists of three different kind of terms

$$\epsilon' Q\langle\mathcal{O}^{(1)}_{z\bar{z}}\rangle = \Omega_1 + \Omega_2 + \Omega_3 + \text{e.o.m.'s} + \text{pure gauge}, \tag{B.25}$$

where we have omitted terms proportional to the ghost equations of motion (3.31) and to the gauge transformations parameterized by $\{\lambda, \bar{\lambda}\}$. We have to impose that the three terms Ω_i vanish separately. The first term is

$$\Omega_1 = (a_3 + a_4 - \bar{a}_3 - \bar{a}_4)\langle\bar{\nabla}(\epsilon\lambda)\nabla(\epsilon'\bar{\lambda})\rangle, \tag{B.26}$$

so we demand

$$a_3 + a_4 = \bar{a}_3 + \bar{a}_4. \tag{B.27}$$

Imposing the vanishing of the second term

$$\begin{aligned}\Omega_2 = {} & \langle(a_1 - \bar{a}_1 + a_3 + a_4 - \bar{a}_3 - \bar{a}_4)[J_3, \epsilon\bar{\lambda}] + (a_2 - \bar{a}_2)[\bar{J}_3, \epsilon'\bar{\lambda}][J_1, \epsilon\lambda] \\ & +(a_1 - \bar{a}_1 + a_2 - \bar{a}_2)[J_2, \epsilon'\lambda][\bar{J}_2, \epsilon\bar{\lambda}]\rangle,\end{aligned} \tag{B.28}$$

we find the additional conditions

$$a_1 = \bar{a}_1, \qquad a_2 = \bar{a}_2. \tag{B.29}$$

Finally, the third term reads

$$\begin{aligned}\Omega_3 = {} & \langle(a_2 + \bar{a}_1)[\bar{J}_1, \epsilon'\lambda][J_1, \epsilon\lambda] + (a_1 + \bar{a}_2)[J_3, \epsilon\bar{\lambda}][\bar{J}_3, \epsilon'\bar{\lambda}] \\ & -a_4[J_3, \epsilon\lambda][\bar{J}_3, \epsilon'\lambda] - \bar{a}_4[\bar{J}_1, \epsilon\bar{\lambda}][J_1, \epsilon'\bar{\lambda}]\rangle.\end{aligned} \tag{B.30}$$

If we expand on the supergroup generators, the first term on the right hand side is proportional to $\lambda^\alpha\lambda^\beta\langle[T_\delta, T_\alpha][T_\rho, T_\beta]\rangle$, where we summarized with a greek letter the various spinor properties of the supercharges and the pure spinors in the various dimensions. In all dimensions, due to the supersymmetry algebra, the term inside the supertrace is proportional to $(\sigma^m)_{\delta\alpha}(\sigma_m)_{\beta\rho}$, where $\sigma^m_{\alpha\beta}$ are the off diagonal blocks of the Dirac matrices. Now comes the crucial property of our lower-dimensional pure spinors, which behave precisely like the ten-dimensional ones. In dimension $d = 2n$, the product of two pure spinors is always proportional to the middle dimensional form $\sigma^{m_1\ldots m_n}_{\alpha\beta}$, therefore the terms in (B.30) are all proportional to

$$\sigma^m \sigma^{m_1\ldots m_n}\sigma_m, \tag{B.31}$$

but this expression vanishes in all even dimensions due to the properties of the gamma matrix algebra. Thus, in all dimensions we find that $\Omega_3 = 0$ identically. As a result, imposing that $\int d^2z\langle\mathcal{O}^{(1)}_{z\bar{z}}\rangle$ is BRST closed requires that the coefficients $a_i, \bar{a}_i$ satisfy (B.27) and (B.29).

On the other hand, the following operator

$$\begin{aligned}\Sigma^{(0)}_{z\bar{z}} = {} & -a_2\bar{J}_2J_2 + (a_1 - a_2)\bar{J}_1J_3 + (a_3 - \bar{a}_4 + a_2 - a_1)N\bar{N} \\ & +(a_4 + a_1 - a_2)w\bar{\nabla}\lambda + (\bar{a}_4 + a_1 - a_2)\bar{w}\nabla\bar{\lambda},\end{aligned} \tag{B.32}$$

is such that

$$Q\int d^2z\langle\Sigma^{(0)}_{z\bar z}\rangle=\int d^2z\langle\mathcal{O}^{(1)}_{z\bar z}\rangle, \tag{B.33}$$

so the cohomology for integrated vertex operators at ghost number one is empty.

C. Supergroups

In this appendix we list the details of the superalgebras we need to realize the various backgrounds in the text. We constructed our superalgebras according to [49].

C.1 Notations

Our notations follow the ones used by [50]. The superalgebra satisfies the following commutation relations:

$$[T_m,T_n]=f^p_{mn}T_p \tag{C.1}$$

$$[T_m,Q_\alpha]=F^\beta_{m\alpha}Q_\beta \tag{C.2}$$

$$\{Q_\alpha,Q_\beta\}=A^m_{\alpha\beta}T_m \tag{C.3}$$

where the T's are the bosonic (Grassman even) generators of a Lie algebra and the Q's are the fermionic (Grassman odd) elements. The indices are $m=1,\ldots,d$ and $\alpha=1,\ldots,D$. The generators satisfy the following super-Jacobi identities:

$$f^p_{nr}f^q_{mp}+f^p_{rm}f^q_{np}+f^p_{mn}f^q_{rp}=0 \tag{C.4}$$

$$F^\gamma_{n\alpha}F^\delta_{m\gamma}-F^\gamma_{m\alpha}F^\delta_{n\gamma}-f^p_{mn}F^\delta_{p\alpha}=0 \tag{C.5}$$

$$F^\delta_{m\gamma}A^n_{\beta\delta}+F^\delta_{m\beta}A^n_{\gamma\delta}-f^n_{mp}A^p_{\beta\gamma}=0 \tag{C.6}$$

$$A^p_{\beta\gamma}F^\delta_{p\alpha}+A^p_{\gamma\alpha}F^\delta_{p\beta}+A^p_{\alpha\beta}F^\delta_{p\gamma}=0 \tag{C.7}$$

Generally we can define a bilinear form

$$<X_M,X_N>=X_MX_N-(-1)^{g(X_M)g(X_N)}X_NX_M=C^P_{NM}X_P \tag{C.8}$$

where X can be either T or Q and $P=1,\ldots,d+D$ (say the first d are T's and the rest D are Q's). $g(X_M)$ is the Grassmann grading, $g(T)=0$ and $g(Q)=1$ and C^P_{NM} are the structure constants. The latter satisfy the graded antisymmetry property

$$C^P_{NM}=-(-1)^{g(X_M)g(X_N)}C^P_{MN} \tag{C.9}$$

We define the super-metric on the super-algebra as the supertrace of the generators in the fundamental representation

$$g_{MN}=\mathrm{Str}X_MX_N, \tag{C.10}$$

We can further define raising and lowering rules when the metric acts on the structure constants

$$C_{MNP}\equiv g_{MS}C^S_{NP} \tag{C.11}$$

$$C_{MNP}=-(-1)^{g(X_N)g(X_P)}C_{MPN}=-(-1)^{g(X_M)g(X_N)}C_{NMP} \tag{C.12}$$

$$C_{MNP}=-(-1)^{g(X_M)g(X_N)+g(X_N)g(X_P)+g(X_P)g(X_M)}C_{PNM} \tag{C.13}$$

For a semi-simple super Lie algebra ($|g_{MN}| \neq 0$ and $|h_{mn}| \neq 0$) we can define a contravariant metric tensor through the relation

$$g_{MP}g^{PN} = \delta_M^N \tag{C.14}$$

The Killing form is defined as the supertrace of the generators in the adjoint representation

$$K_{MN} \equiv (-1)^{g(X_P)} C^S_{PM} C^P_{SN} = (-1)^{g(X_M)g(X_N)} K_{NM} \tag{C.15}$$

(while on the (sub)Lie-algebra we define the metric $K_{mn} = f^p_{mq} f^q_{np}$). Explicitly we have

$$K_{mn} = h_{mn} - F^\beta_{m\alpha} F^\alpha_{n\beta} = K_{nm} \tag{C.16}$$

$$K_{\alpha\beta} = F^\gamma_{m\alpha} A^m_{\beta\gamma} - F^\gamma_{m\beta} A^m_{\alpha\gamma} = -K_{\beta\alpha} \tag{C.17}$$

$$K_{m\alpha} = K_{\alpha m} = 0 \tag{C.18}$$

The Killing form is proportional to the supermetric up to the second Casimir $C_2(G)$ of the supergroup, which is also called the dual Coxeter number

$$K_{MN} = -\, C_2(G)\, g_{MN}. \tag{C.19}$$

In the main text, we have computed the one-loop beta-functions in the background field method. It turns out that the sums of one-loop diagrams with fixed external lines are proportional to the Ricci tensor R_{MN} of the supergroup. The super Ricci tensor of a supergroup is defined as

$$R_{MN}(G) = -\frac{1}{4} f^P_{MQ} f^Q_{NP} (-)^{g(X_Q)}, \tag{C.20}$$

and we immediately see that $R_{MN} = -K_{MN}$, in particular, we can write it as

$$R_{MN}(G) = \frac{C_2(G)}{4} g_{MN}, \tag{C.21}$$

C.2 Summary of our models

We would like to have a clear spacetime interpretation of the dual Coxeter number of a supergroup. Let us consider a supergroup G with a $\mathbb{Z}_4$ automorphism, whose zero locus we denote by H. The various RR backgrounds we discussed in the main text are realized as G/H supercosets of this kind. The bosonic submanifold is in general $AdS_p \times S^q$, where the gauge group $H = \mathrm{SO}(1, p-1) \times \mathrm{SO}(q) \times \mathrm{SO}(r)$, and the $\mathrm{SO}(r)$ factor corresponds to the non-geometric isometries. We have the following cases

	G	Algebra	p	q	r	$\#_{\text{susy}}$	$C_2(G)$
AdS_2	$Osp(1\|2)$	$B(0\|1)$	2	0	0	2	-3
AdS_2	$Osp(2\|2)$	$C(2)$	2	0	2	4	-2
AdS_4	$Osp(2\|4)$	$C(3)$	4	0	2	8	-4
AdS_6	F_4	$F(4;3)$	6	0	3	16	2
$AdS_5 \times S^1$	$SU(2,2\|2)$	$A(2\|4)$	5	0	3	16	4
$AdS_2 \times S^2$	$PSU(1,1\|2)$	$A(2\|2)$	2	2	0	8	0
$AdS_3 \times S^3$	$PSU(1,1\|2)^2$	$A(2\|2) \oplus A(2\|2)$	3	3	0	16	0
$AdS_5 \times S^5$	$PSU(2,2\|4)$	$A(4\|4)$	5	5	0	32	0

The superspace notations will be as follows: the letters $\{M, N, \ldots\}$ refer to elements of the supergroup G, while $\{I, J, \ldots\}$ take values in the gauge group H and finally $\{A, B, \ldots\}$ refer to elements of the supercoset G/H. The lower case letters denote the bosonic and fermionic components of the superspace indices, while $\#_{\rm susy}$ is the number of real space-time supercharges in the background. Then, we can rewrite the super Ricci tensor of the supergroup (C.20) making explicit the $\mathbb{Z}_4$ grading[28]

$$R_{AB}(G) = -\frac{1}{4} f^C_{AD} f^D_{BC} (-)^C - \frac{1}{2} f^I_{AD} f^D_{BI} (-)^I, \tag{C.22}$$

In particular, its grading two part is

$$R_{ab}(G) = \frac{1}{4}\left(F^{\alpha}_{a\hat{\beta}} F^{\hat{\beta}}_{b\alpha} + F^{\hat{\alpha}}_{a\alpha} F^{\alpha}_{b\hat{\alpha}}\right) - \frac{1}{2} f^i_{ac} f^c_{bi}, \tag{C.23}$$

C.3 $Osp(2|2)$

The Osp(2|2) supergroup corresponds to the superalgebra $C(2)$. It has a bosonic subgroup $\mathrm{Sp}(2) \times \mathrm{SO}(2)$ and four real fermionic generators transforming in the $\mathbf{4} \oplus \mathbf{4}$ of Sp(2). It consists of the super matrices $\mathbf{M}$ satisfying $\mathbf{M}^{\mathbf{st}}\mathbf{H}\mathbf{M} = \mathbf{H}$, where

$$\mathbf{H} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The superalgebra is obtained by the commutation relations $\mathbf{m}^{\mathbf{st}}\mathbf{H} + \mathbf{H}\mathbf{m} = 0$, where we parameterize

$$\mathbf{m} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{sl(2)} & a & b \\ & & c & d \\ \hline e & f & & \\ g & h & \multicolumn{2}{c}{so(2)} \end{array}\right) \qquad \mathbf{m}^{\mathbf{st}} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{sl(2)^t} & e & g \\ & & f & h \\ \hline -a & -c & & \\ -c & -d & \multicolumn{2}{c}{so(2)^t} \end{array}\right) \tag{C.24}$$

so that from the condition $\mathbf{m}^{\mathbf{st}}\mathbf{H} + \mathbf{H}\mathbf{m} = 0$ we find

$$\mathbf{m} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{sl(2)} & a & b \\ & & c & d \\ \hline -c & a & & \\ -d & b & \multicolumn{2}{c}{so(2)} \end{array}\right).$$

The Cartan basis for the $Osp(2|2)$ superalgebra is given by the following supermatrices. The bosonic generators are

$$\mathbf{H} = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & -1 & & \\ \hline & & & \\ & & & \end{array}\right), \quad \mathbf{E}^+ = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & & \\ & & & \end{array}\right), \quad \mathbf{E}^- = \left(\begin{array}{cc|cc} 0 & 0 & & \\ 1 & 0 & & \\ \hline & & & \\ & & & \end{array}\right), \quad \widetilde{\mathbf{H}} = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & 0 & 1 \\ & & -1 & 0 \end{array}\right),$$

[28]The Ricci tensor of the supercoset G/H is given $R_{AB}(G/H) = -\frac{1}{4} f^C_{AD} f^D_{BC}(-)^C - f^I_{AD} f^D_{BI}(-)^I$, see [29].

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where $(\mathbf{H}, \mathbf{E}^{\pm})$ are the generators of $sl(2)$ while $\widetilde{\mathbf{H}}$ is the generator of SO(2). The fermionic generators $(\mathbf{Q}_{\alpha}, \mathbf{Q}_{\hat{\alpha}})$ are

$$\mathbf{Q_1} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 0 & & \\ 0 & 1 & & \end{array}\right), \qquad \mathbf{Q_{\hat{1}}} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & -1 \\ \hline 0 & 0 & & \\ 1 & 0 & & \end{array}\right),$$

$$\mathbf{Q_2} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 0 \\ \hline 0 & 1 & & \\ 0 & 0 & & \end{array}\right) \qquad \mathbf{Q_{\hat{2}}} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\begin{array}{cc|cc} & & 0 & 0 \\ & & -1 & 0 \\ \hline 1 & 0 & & \\ 0 & 0 & & \end{array}\right)$$

Finally, the $Osp(2|2)$ superalgebra is given by

$$\begin{aligned}
&[\mathbf{H}, \mathbf{E}^{\pm}] = \pm 2\mathbf{E}^{\pm}, \qquad [\mathbf{E}^{+}, \mathbf{E}^{-}] = \mathbf{H}, \qquad [\mathbf{H}, \widetilde{\mathbf{H}}] = 0, \\
&[\widetilde{\mathbf{H}}, \mathbf{E}^{\pm}] = 0, \qquad [\widetilde{\mathbf{H}}, \mathbf{Q}_{\alpha}] = \epsilon_{\alpha\beta}\mathbf{Q}_{\beta}, \qquad [\widetilde{\mathbf{H}}, \mathbf{Q}_{\hat{\alpha}}] = \epsilon_{\hat{\alpha}\hat{\beta}}\mathbf{Q}_{\hat{\beta}}, \\
&[\mathbf{H}, \mathbf{Q}_{\alpha}] = \mathbf{Q}_{\alpha}, \qquad [\mathbf{H}, \mathbf{Q}_{\hat{\alpha}}] = -\mathbf{Q}_{\hat{\alpha}}, \\
&\{\mathbf{Q}_{\alpha}, \mathbf{Q}_{\beta}\} = \frac{1}{2}\delta_{\alpha\beta}\mathbf{E}^{+}, \quad \{\mathbf{Q}_{\hat{\alpha}}, \mathbf{Q}_{\hat{\beta}}\} = \frac{1}{2}\delta_{\hat{\alpha}\hat{\beta}}\mathbf{E}^{-}, \quad \{\mathbf{Q}_{\alpha}, \mathbf{Q}_{\hat{\alpha}}\} = \frac{1}{2}\delta_{\alpha\hat{\alpha}}\mathbf{H} + \frac{1}{2}\epsilon_{\alpha\hat{\alpha}}\widetilde{\mathbf{H}}, \\
&[\mathbf{E}^{+}, \mathbf{Q}_{\alpha}] = [\mathbf{E}^{-}, \mathbf{Q}_{\hat{\alpha}}] = 0, \qquad [\mathbf{E}^{+}, \mathbf{Q}_{\hat{\alpha}}] = -\delta_{\hat{\alpha}\alpha}\mathbf{Q}_{\alpha}, \qquad [\mathbf{E}^{-}, \mathbf{Q}_{\alpha}] = -\delta_{\alpha\hat{\alpha}}\mathbf{Q}_{\hat{\alpha}},
\end{aligned} \tag{C.25}$$

We classify the generators according to their $\mathbb{Z}_4$ charge

$$\begin{array}{c|c|c|c} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\ & & & \\ \mathbf{H}, \widetilde{\mathbf{H}} & \mathbf{Q}_{\alpha} & \mathbf{E}^{\pm} & \mathbf{Q}_{\hat{\alpha}}. \end{array} \tag{C.26}$$

In the main text, we realize our AdS_2 background by quotienting with respect to the grading zero subgroup, namely $SO(1,1) \times SO(2)$. The structure constants are

$$\begin{aligned}
f^{H}_{ml} &= \delta^{+}_{m}\delta^{-}_{l} - \delta^{-}_{m}\delta^{+}_{l}, & f^{\widetilde{H}}_{ml} &= 0, \\
f^{l}_{Hm} &= 2(\delta^{+}_{m}\delta^{l}_{+} - \delta^{-}_{m}\delta^{l}_{-}), & f^{l}_{\widetilde{H}m} &= 0 \\
F^{\hat{\alpha}}_{\alpha m} &= \delta^{\hat{\alpha}}_{\alpha}\delta^{-}_{m}, & F^{\alpha}_{\hat{\alpha}m} &= \delta^{\alpha}_{\hat{\alpha}}\delta^{+}_{m} \\
F^{\beta}_{H\alpha} &= \delta^{\beta}_{\alpha}, & F^{\hat{\beta}}_{H\hat{\alpha}} &= -\delta^{\hat{\beta}}_{\hat{\alpha}} \\
F^{\beta}_{\widetilde{H}\alpha} &= \epsilon_{\alpha\gamma}\delta^{\gamma\beta}, & F^{\hat{\beta}}_{\widetilde{H}\hat{\alpha}} &= \epsilon_{\hat{\alpha}\hat{\gamma}}\delta^{\hat{\gamma}\hat{\beta}} \\
A^{m}_{\alpha\beta} &= \delta_{\alpha\beta}\delta^{m}_{+}, & A^{m}_{\hat{\alpha}\hat{\beta}} &= -\delta_{\hat{\alpha}\hat{\beta}}\delta^{m}_{-}, \\
A^{H}_{\alpha\hat{\beta}} &= A^{H}_{\hat{\beta}\alpha} = \frac{1}{2}\delta_{\alpha\hat{\beta}}, & A^{\widetilde{H}}_{\alpha\hat{\beta}} &= A^{\widetilde{H}}_{\hat{\beta}\alpha} = \frac{1}{2}\epsilon_{\alpha\hat{\beta}},
\end{aligned}$$

The metric on the supergroup is

$$\begin{aligned}
g_{mn} &= \delta^{+}_{m}\delta^{-}_{n} + \delta^{-}_{m}\delta^{+}_{n}, \\
g_{\alpha\hat{\alpha}} &= -\eta_{\hat{\alpha}\alpha} = \delta_{\alpha\hat{\alpha}}, \qquad g_{ij} = 2\delta_{ij},
\end{aligned} \tag{C.27}$$

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where $m,n=\pm$, $i,j=H,\widetilde{H}$.

The $OSp(1|2)$ supergroup corresponds to the superalgebra $B(0|1)$. Its bosonic subgroup is Sp(2) and it has two real fermionic generators transforming in the **2** of Sp(2). It can be easily obtained by the one of the $Osp(2|2)$ supergroup by simply dropping the generators $\widetilde{\mathbf{H}}$ and $\mathbf{Q}_2, \mathbf{Q}_{\hat{2}}$.

C.4 $Osp(2|4)$

The supergroup $Osp(2|4)$ corresponds to the superalgebra $C(3)$. Its bosonic subgroup is $\mathrm{Sp}(4)\times\mathrm{SO}(2)$ and it has eight real fermionic generators transforming in the $\mathbf{4}\oplus\mathbf{4}$ of Sp(4). We classify the generators according to their $\mathbb{Z}_4$ charge

$$\begin{array}{c|c|c|c} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\ \\ \mathbf{J}_{[\mathbf{ab}]}, \widetilde{\mathbf{H}} & \mathbf{Q}_\alpha & \mathbf{P}_a & \mathbf{Q}_{\hat{\alpha}}, \end{array} \tag{C.28}$$

where $a=0,\ldots,3$ and $\alpha,\hat{\alpha}$ are four-dimensional Majorana spinor indices. In the main text, we realize our AdS_4 background by quotienting with respect to the grading zero subgroup, namely $\mathrm{SO}(1,3)\times\mathrm{SO}(2)$. The structure constants are

$$\begin{aligned}
f_{ab}^{[cd]} &= \frac{1}{2}\delta_a^{[c}\delta_b^{d]}, & f_{a[bc]}^{d} &= -f_{[bc]a}^{d} = \eta_{a[b}\delta_{c]}^{d} \\
f_{[ab][cd]}^{[ef]} &= \frac{1}{2}\delta_{[a}^{[e}\eta_{b][c}\delta_{d]}^{f]} = \frac{1}{2}\Big(\eta_{bc}\delta_a^{[e}\delta_d^{f]} + \eta_{ad}\delta_b^{[e}\delta_c^{f]} - \eta_{ac}\delta_b^{[e}\delta_d^{f]} - \eta_{bd}\delta_a^{[e}\delta_c^{f]}\Big) \\
F_{a\alpha}^{\widehat{\beta}} &= -F_{\alpha a}^{\widehat{\beta}} = \frac{1}{2}(\gamma_a)^{\beta}{}_{\alpha}\delta^{\widehat{\beta}}{}_{\beta}, & F_{a\hat{\alpha}}^{\beta} &= -F_{\hat{\alpha}a}^{\beta} = \frac{1}{2}(\gamma_a)^{\widehat{\beta}}{}_{\hat{\alpha}}\delta^{\beta}{}_{\widehat{\beta}} \\
F_{[ab]\alpha}^{\beta} &= -F_{\alpha[ab]}^{\beta} = \frac{1}{2}(\gamma_{ab})^{\beta}{}_{\alpha}, & F_{[ab]\hat{\alpha}}^{\widehat{\beta}} &= -F_{\hat{\alpha}[ab]}^{\widehat{\beta}} = \frac{1}{2}(\gamma_{ab})^{\widehat{\beta}}{}_{\hat{\alpha}} \\
F_{\widetilde{H}\alpha}^{\beta} &= \frac{1}{2}(\gamma^5)_{\alpha}{}^{\beta}, & F_{\widetilde{H}\hat{\alpha}}^{\widehat{\beta}} &= -\frac{1}{2}(\gamma^5)_{\hat{\alpha}}{}^{\widehat{\beta}} \\
A_{\alpha\beta}^{a} &= (C\gamma^a)_{\alpha\beta}, & A_{\hat{\alpha}\widehat{\beta}}^{a} &= (C\gamma^a)_{\hat{\alpha}\widehat{\beta}} \\
A_{\alpha\widehat{\beta}}^{\widetilde{H}} &= -2(\gamma^5)_{\alpha}{}^{\gamma}(\widetilde{C})_{\gamma\widehat{\beta}}, & A_{\hat{\alpha}\beta}^{\widetilde{H}} &= 2(\gamma^5)_{\hat{\alpha}}{}^{\widehat{\gamma}}(\widetilde{C})_{\widehat{\gamma}\beta} \\
A_{\alpha\widehat{\beta}}^{[ab]} &= -\frac{1}{2}(\widetilde{C})_{\alpha\widehat{\gamma}}(\gamma^{ab})^{\widehat{\gamma}}{}_{\widehat{\beta}}, & A_{\hat{\alpha}\beta}^{[ab]} &= -\frac{1}{2}(\widetilde{C})_{\hat{\alpha}\gamma}(\gamma^{ab})^{\gamma}{}_{\beta}
\end{aligned} \tag{C.29}$$

where C is the charge conjugation matrix of SO(1,3). The supermetric is given by

$$\begin{aligned}
g_{ab} &= \eta_{ab}, & g_{\alpha\widehat{\beta}} &= 2C_{\alpha\widehat{\beta}} \\
g_{[ab][cd]} &= \eta_{a[c}\eta_{d]b}, & g_{\widetilde{H}\widetilde{H}} &= 2.
\end{aligned} \tag{C.30}$$

C.5 SU(2,2|2)

The supergroup SU(2,2|2) corresponds to the superalgebra $A(3|1)$. Its bosonic subgroup is $\mathrm{SU}(2,2)\times\mathrm{SO}(3)\times\mathrm{U}(1)$ and it has sixteen real fermionic generators transforming in the $(\bar{\mathbf{4}},\mathbf{2})\oplus(\mathbf{4},\bar{\mathbf{2}})$. We classify the generators according to their $\mathbb{Z}_4$ charge

$$\begin{array}{c|c|c|c} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\ \\ \mathbf{J}_{[\mathbf{ab}]}, \widetilde{\mathbf{H}}_{\mathbf{a}'} & \mathbf{Q}_{\alpha\alpha'} & \mathbf{P}_a, \mathbf{R} & \mathbf{Q}_{\hat{\alpha}\hat{\alpha}'}, \end{array} \tag{C.31}$$

where $a = 0, \ldots, 4$ are the coordinates on the AdS_5, $\mathbf{R}$ is the translation generator on S^1 and $a' = 1, 2, 3$ is the SO(3) vector index, while $(\alpha, \hat{\alpha})$ are Majorana spinor indices of SO(1, 4) and $(\alpha', \hat{\alpha}')$ are spinors of SO(3). In the main text, we realize our $AdS_5 \times S^1$ background by quotienting with respect to the grading zero subgroup.

Its structure constants are

$$
\begin{aligned}
f_{ab}^{[cd]} &= \frac{1}{2}\delta_a^{[c}\delta_b^{d]}, \qquad f_{a'b'}^{c'} = \epsilon_{a'b'}{}^{c'}, \quad f_{a[bc]}^{d} = -f_{[bc]a}^{d} = \eta_{a[b}\delta_{c]}^{d} \\
f_{[ab][cd]}^{[ef]} &= \frac{1}{2}\delta_{[a}^{[e}\eta_{b][c}\delta_{d]}^{f]} = \frac{1}{2}\left(\eta_{bc}\delta_a^{[e}\delta_d^{f]} + \eta_{ad}\delta_b^{[e}\delta_c^{f]} - \eta_{ac}\delta_b^{[e}\delta_d^{f]} - \eta_{bd}\delta_a^{[e}\delta_c^{f]}\right) \\
F_{\alpha\alpha' a}^{\widehat{\beta}\widehat{\beta}'} &= -\frac{i}{2}(\gamma_a)^{\widehat{\beta}}{}_{\alpha}\delta^{\widehat{\beta}'}{}_{\alpha'}, \qquad F_{\hat{\alpha}\hat{\alpha}' a}^{\beta\beta'} = \frac{i}{2}(\gamma_a)^{\beta}{}_{\hat{\alpha}}\delta^{\beta'}{}_{\hat{\alpha}'} \\
F_{\alpha\alpha' R}^{\widehat{\beta}\widehat{\beta}'} &= \frac{1}{2}\delta^{\widehat{\beta}'}{}_{\alpha'}\delta^{\widehat{\beta}}{}_{\alpha}, \qquad F_{\hat{\alpha}\hat{\alpha}' R}^{\beta\beta'} = -\frac{1}{2}\delta^{\beta'}{}_{\hat{\alpha}'}\delta^{\beta}{}_{\hat{\alpha}} \\
F_{\alpha\alpha'[ab]}^{\beta\beta'} &= -\frac{1}{2}(\gamma_{ab})^{\beta}{}_{\alpha}\delta^{\beta'}{}_{\alpha'}, \qquad F_{\hat{\alpha}\hat{\alpha}'[ab]}^{\widehat{\beta}\widehat{\beta}'} = -\frac{1}{2}(\gamma_{ab})^{\widehat{\beta}}{}_{\hat{\alpha}}\delta^{\widehat{\beta}'}{}_{\hat{\alpha}'} \\
F_{\alpha\alpha' a'}^{\beta\beta'} &= -\frac{1}{2}(\tau_{a'})^{\beta'}{}_{\alpha'}\delta^{\beta}{}_{\alpha}, \qquad F_{\hat{\alpha}\hat{\alpha}' a'}^{\widehat{\beta}\widehat{\beta}'} = -\frac{1}{2}(\tau_{a'})^{\widehat{\beta}'}{}_{\hat{\alpha}'}\delta^{\widehat{\beta}}{}_{\hat{\alpha}} \\
A_{\alpha\alpha'\beta\beta'}^{a} &= -iC_{\alpha'\beta'}(C\gamma^a)_{\alpha\beta}, \qquad A_{\hat{\alpha}\hat{\alpha}'\widehat{\beta}\widehat{\beta}'}^{a} = -iC_{\hat{\alpha}'\widehat{\beta}'}(C\gamma^a)_{\hat{\alpha}\widehat{\beta}} \\
A_{\alpha\alpha'\beta\beta'}^{R} &= C_{\alpha\beta}C'_{\alpha'\beta'}, \qquad A_{\hat{\alpha}\hat{\alpha}'\widehat{\beta}\widehat{\beta}'}^{R} = C_{\hat{\alpha}\widehat{\beta}}C'_{\hat{\alpha}'\widehat{\beta}'} \\
A_{\alpha\alpha'\widehat{\beta}\widehat{\beta}'}^{[ab]} &= \frac{1}{2}C_{\alpha'\widehat{\beta}'}(C\gamma^{ab})_{\alpha\widehat{\beta}}, \qquad A_{\hat{\alpha}\hat{\alpha}'\beta\beta'}^{[ab]} = -\frac{1}{2}C_{\hat{\alpha}'\beta'}(C\gamma^{ab})_{\hat{\alpha}\beta} \\
A_{\alpha\alpha'\widehat{\beta}\widehat{\beta}'}^{a'} &= -2C_{\alpha\widehat{\beta}}(C'\tau^{a'})_{\alpha'\widehat{\beta}'}, \qquad A_{\hat{\alpha}\hat{\alpha}'\beta\beta'}^{a'} = 2C_{\hat{\alpha}\beta}(C'\tau^{a'})_{\hat{\alpha}'\beta'},
\end{aligned}
\tag{C.32}
$$

where $C_{\alpha\beta}$ and $C'_{\alpha'\beta'}$ are the charge conjugation matrices respectively of SO(1, 4) and SO(3). The supermetric is

$$
\begin{aligned}
g_{ab} &= -\eta_{ab}, \qquad g_{RR} = 1, \\
g_{\alpha\widehat{\beta}} &= C_{\alpha\widehat{\beta}}, \qquad g_{\alpha'\widehat{\beta}'} = C'_{\alpha'\widehat{\beta}'}.
\end{aligned}
\tag{C.33}
$$

C.6 $AdS_p \times S^p$ superalgebras

The $AdS_p \times S^p$ backgrounds are realized by the following supercosets

$$
\begin{array}{ccc}
AdS_2 \times S^2 & AdS_3 \times S^3 & AdS_5 \times S^5 \\
\frac{PSU(1,1|2)}{\mathrm{SO}(1,1)\times\mathrm{SO}(2)} & \frac{PSU(1,1|2)^2}{\mathrm{SO}(1,2)\times\mathrm{SO}(3)} & \frac{PSU(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}
\end{array}
\tag{C.34}
$$

We can treat the supergroups $PSU(1,1|2)$, $PSU(1,1|2)^2$ and $PSU(2,2|4)$ schematically altogether, by collecting their generators according to their $\mathbb{Z}_4$ grading as follows

$$
\begin{array}{c|c|c|c}
\mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 \\
 & & & \\
\mathbf{J}_{[\mathbf{ab}]}, \mathbf{J}_{\mathbf{a'b'}} & \mathbf{Q}_{\alpha\alpha'} & \mathbf{P}_a, \mathbf{P}_{a'} & \mathbf{Q}_{\hat{\alpha}\hat{\alpha}'},
\end{array}
\tag{C.35}
$$

where $a = 0, \ldots, p-1$ are the coordinates on the AdS_p, $a' = 1, \ldots, p$ are the coordinates along S^p, while $(\alpha, \hat{\alpha})$ are Weyl spinor indices of SO(1, $p-1$) and $(\alpha', \hat{\alpha}')$ are spinors of

SO(p). In the main text, we realize our $AdS_p \times S^p$ backgrounds by quotienting with respect to the grading zero subgroup. The structure constants are

$$f_{ab}^{[cd]} = \frac{1}{2}\delta_a^{[c}\delta_b^{d]} \qquad f_{a'b'}^{[c'd']} = -\frac{1}{2}\delta_{a'}^{[c'}\delta_b^{\prime d']} \qquad f_{\underline{a}[\underline{bc}]}^{\underline{d}} = -f_{[\underline{bc}]\underline{a}}^{\underline{d}} = \eta_{\underline{a}[\underline{b}}\delta_{\underline{c}]}^{\underline{d}} \tag{C.36}$$

$$f_{[\underline{ab}][\underline{cd}]}^{[\underline{ef}]} = \frac{1}{2}\delta_{[\underline{a}}^{[\underline{e}}\eta_{\underline{b}][\underline{c}}\delta_{\underline{d}]}^{\underline{f}]} = \frac{1}{2}\left(\eta_{\underline{bc}}\delta_{\underline{a}}^{[\underline{e}}\delta_{\underline{d}}^{\underline{f}]} + \eta_{\underline{ad}}\delta_{\underline{b}}^{[\underline{e}}\delta_{\underline{c}}^{\underline{f}]} - \eta_{\underline{ac}}\delta_{\underline{b}}^{[\underline{e}}\delta_{\underline{d}}^{\underline{f}]} - \eta_{\underline{bd}}\delta_{\underline{a}}^{[\underline{e}}\delta_{\underline{c}}^{\underline{f}]}\right)$$

$$\begin{aligned}
F_{\alpha\alpha' a}^{\widehat{\beta}\widehat{\beta}'} &= -\frac{i}{2}(\gamma_a)^{\widehat{\beta}}{}_{\alpha}\delta^{\widehat{\beta}'}{}_{\alpha'}, & F_{\hat{\alpha}\hat{\alpha}' a}^{\beta\beta'} &= \frac{i}{2}(\gamma_a)^{\beta}{}_{\hat{\alpha}}\delta^{\beta'}{}_{\hat{\alpha}'} \\
F_{\alpha\alpha' a'}^{\widehat{\beta}\widehat{\beta}'} &= \frac{1}{2}(\gamma_{a'})^{\widehat{\beta}'}{}_{\alpha'}\delta^{\widehat{\beta}}{}_{\alpha}, & F_{\hat{\alpha}\hat{\alpha}' a'}^{\beta\beta'} &= -\frac{1}{2}(\gamma_{a'})^{\beta'}{}_{\hat{\alpha}'}\delta^{\beta}{}_{\hat{\alpha}} \\
F_{\alpha\alpha'[\underline{ab}]}^{\beta\beta'} &= -\frac{1}{2}(\gamma_{\underline{ab}})^{\beta}{}_{\alpha}\delta^{\beta'}{}_{\alpha'}, & F_{\hat{\alpha}\hat{\alpha}'[\underline{ab}]}^{\widehat{\beta}\widehat{\beta}'} &= \frac{1}{2}(\gamma_{\underline{ab}})^{\widehat{\beta}}{}_{\hat{\alpha}}\delta^{\widehat{\beta}'}{}_{\hat{\alpha}'} \\
A_{\alpha\alpha'\beta\beta'}^{a} &= -iC_{\alpha'\beta'}(C\gamma^a)_{\alpha\beta}, & A_{\hat{\alpha}\hat{\alpha}'\widehat{\beta}\widehat{\beta}'}^{a} &= -iC_{\hat{\alpha}'\widehat{\beta}'}(C\gamma^a)_{\hat{\alpha}\widehat{\beta}} \\
A_{\alpha\alpha'\beta\beta'}^{a'} &= C_{\alpha\beta}(C'\gamma^{a'})_{\alpha'\beta'}, & A_{\hat{\alpha}\hat{\alpha}'\widehat{\beta}\widehat{\beta}'}^{a'} &= C_{\hat{\alpha}\widehat{\beta}}(C'\gamma^{a'})_{\hat{\alpha}'\widehat{\beta}'} \\
A_{\alpha\alpha'\widehat{\beta}\widehat{\beta}'}^{[ab]} &= \frac{1}{2}C_{\alpha'\widehat{\beta}'}(C\gamma^{ab})_{\alpha\widehat{\beta}}, & A_{\hat{\alpha}\hat{\alpha}'\beta\beta'}^{[ab]} &= -\frac{1}{2}C_{\hat{\alpha}'\beta'}(C\gamma^{ab})_{\hat{\alpha}\beta} \\
A_{\alpha\alpha'\widehat{\beta}\widehat{\beta}'}^{[a'b']} &= -\frac{1}{2}C_{\alpha\widehat{\beta}}(C'\gamma^{a'b'})_{\alpha'\widehat{\beta}'}, & A_{\hat{\alpha}\hat{\alpha}'\beta\beta'}^{[a'b']} &= \frac{1}{2}C_{\hat{\alpha}\beta}(C'\gamma^{a'b'})_{\hat{\alpha}'\beta'}
\end{aligned} \tag{C.37}$$

where $C_{\alpha\beta}$ and $C'_{\alpha'\beta'}$ are respectively the charge conjugation matrices of SO$(1, p-1)$ and SO(p). The supermetric in the fundamental is

$$g_{ab} = -\eta_{ab}, \qquad g_{a'b'} = \eta_{a'b'}, \tag{C.38}$$

$$g_{\alpha\hat{\alpha}} = C_{\alpha\hat{\alpha}}C'_{\alpha'\hat{\alpha}'}, \tag{C.39}$$

D. Non-critical supergravity in d dimensions

In this section we study d-dimensional supergravity with a cosmological constant. We will show that there is an $AdS_{d-1} \times S^1$ solution with RR $(d-1)$-form flux only when we introduce space-filling sources, that can be interpreted as uncharged D_{d-1}-branes. This reproduces the results found in [19] for the $AdS_5 \times S^1$ case and gives the new solution $AdS_3 \times S^1$. In the next section we will then argue that, by including the first α' corrections to the non-critical supergravity equations, an $AdS_5 \times S^1$ solution might be possible, even without the space-filling sources.

The d-dimensional non-critical supergravity action in the string frame is

$$S = \frac{1}{\kappa^2}\int d^dx\,\sqrt{-G}\left[-(\partial_\mu\chi)^2 + e^{-2\phi}\left(R + 4(\partial_\mu\phi) + \Lambda\right) - 2N_f e^{-\phi}\right], \tag{D.1}$$

where χ is the RR scalar, dual to the $(d-1)$-form flux, the cosmological constant is

$$\Lambda = \frac{10-d}{\alpha'}, \tag{D.2}$$

and the last term is the contribution of N_f pairs of space-filling uncharged sources. In [19] this term was interpreted as arising from N_f pairs of branes and anti-branes. We make an ansatz for a solution of the kind $AdS_{d-1} \times S^1$ with constant dilaton. The equations of motion for the metric and the dilaton then reduce to

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{D-2} G_{\mu\nu} \left(2N_f e^{\phi} - \Lambda \right) + e^{2\phi} \partial_\mu \chi \partial_\nu \chi, \\
0 &= R + \Lambda - N_f e^{\phi},
\end{aligned}
\tag{D.3}
$$

Let us parameterize the dual of the RR $(d-1)$-form flux as a one-form flux $\partial_\mu \chi$, whose only nonzero leg is along the S^1 as in the previous section. In particular, $\chi \sim N_c \theta$, where θ is the coordinate on the circle, so that $\partial_\theta \chi \sim N_c$. Note that component of the Ricci curvature along the circle vanishes $R_{\theta\theta} = 0$. Then we find the following solution for the scalar curvature and the string coupling

$$
R = -\frac{10-d}{d}(d-1), \qquad g_S \equiv e^{\phi} = \frac{10-d}{d} N_f, \tag{D.4}
$$

and the components of the Ricci curvature along the AdS_{d-1} are $R_{ij} = -\frac{10-d}{d} G_{ij}$. Recalling that $R_{ij} = -\frac{d-2}{R^2_{\text{AdS}}} G_{ij}$ and $G_{\theta\theta} = R_S^2$ we can read off the radii

$$
R^2_{\text{AdS}} = \frac{d(d-2)}{10-d} \alpha', \qquad R_S^2 = \frac{10-d}{d} \frac{N_c^2}{N_f^2}. \tag{D.5}
$$

It is easy to see that, without space-filling sources, namely if we set $N_f = 0$, then there is no solution to the supergravity equations (D.3). In the case $d = 6$ we recover the $AdS_5 \times S^1$ solution of [19]. Moreover, we find the new solution

$$
AdS_3 \times S^1, \qquad R^2_{\text{AdS}} = \frac{4}{3} \alpha', \qquad R_S^2 = \frac{3}{2} \frac{N^2}{N_f^2} \alpha'. \tag{D.6}
$$

It would be interesting to repeat the analysis of [19] for this last case, to understand its relation to the four-dimensional type II linear dilaton background.

D.1 Higher curvature corrections

We have just seen above that there is no $AdS_5 \times S^1$ solution to the six-dimensional non-critical supergravity equations, unless we include space-filling sources. We will now argue that, when we include the first α' corrections to the supergravity equations, there might be a solution *without* the space-filling sources. At the end of the section, we will comment about the validity of our argument.

We use the methods of [42] and [51]. We make the ansatz for a solution of non-critical supergravity of the form $AdS_5 \times S^1$ with constant dilaton $g_s = e^{\phi}$ and constant RR five-form flux F_5. We are not adding any space-filling brane. Let us denote by R_{AdS}, R_S the radii of AdS and S^1 respectively and parameterize the dual of the RR five-form flux as a one form flux $\partial_\mu \chi$, whose only nonzero leg is along the S^1 as in the previous section, in particular $\chi \sim N_c \theta$. If we plug these ansatze in the ordinary supergravity action (D.1), we find the leading order terms

$$
S_0 = V g_s^{-2} R^2_{\text{AdS}} R_S \left(-\frac{20}{R^2_{\text{AdS}}} + \Lambda - \frac{(g_s N)^2}{R_S^2} \right), \tag{D.7}
$$

where $\Lambda = (10-d)/\alpha'$. Note that all the components of the Riemann tensor along the circle direction vanish, so that we do not have a term proportional to R_S^{-2}, which would come from the ordinary Einstein-Hilbert action.

Let us address now the first α' corrections to the supergravity action, which are basically of three kinds. The curvature squared terms of the kind $R^2_{\mu\nu\rho\sigma}, R^2_{\mu\nu}, R^2$ are all proportional to $R^{-4}_{\rm AdS}$. The fourth power of the RR field strength

$$(g^{\mu\nu}\partial_\mu\chi\partial\nu\chi)^2 \sim \frac{(g_sN_c)^4}{R_S^4}, \tag{D.8}$$

while the only mixed coupling between the RR field strength and the curvatures is

$$R(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi) \sim \frac{(g_sN_c)^2}{R^2_{\rm AdS}R_S^2}. \tag{D.9}$$

Collecting all the terms we find the first α' corrections to the action

$$S_1 = Vg_s^{-2}R^5_{\rm AdS}R_S\left(\frac{\gamma}{R^4_{\rm AdS}} + A\frac{(g_sN_c)^2}{R^2_{\rm AdS}R_S^2} + B\frac{(g_sN_c)^4}{R_S^4}\right), \tag{D.10}$$

where γ, A, B are real coefficients.

The goal now is to vary the action S_0+S_1 with respect to $R_{\rm AdS}, R_S, g_s$ and look for a real positive solution for some values of the coefficients A, B, γ. The three variations of the action read

$$\begin{aligned} \frac{1}{2B}\left(\frac{s}{a}\right)^2\left(-40a+8a^2+2\gamma\right) &= g^2,\\ -20s^2a+4s^2a^2+gsa^2+\gamma s^2-Agsa-3Bg^2a^2 &= 0,\\ -60as^2+20a^2s^2-5a^2sg+\gamma s^2+3Agas+5Ba^2g^2 &= 0. \end{aligned} \tag{D.11}$$

where

$$a \equiv R^2_{\rm AdS}, \quad s \equiv R_S^2, \quad g \equiv (g_sN)^2.$$

One can first check that if $A, B = 0$ there is no solution to the equations of motion. This is the result of [51] that, if we include only the curvature squared terms and not the RR couplings, there is still no $AdS_5 \times S^1$ solution.

It turns out that there is a solution with nonvanishing coefficients γ, A, B, for real positive radii squared and string coupling, namely

$$\begin{aligned} g &= 1,\\ s &= \frac{a}{4(2a-5)},\\ a &= \frac{5}{2}+\frac{1}{2}\sqrt{\frac{\gamma-25}{16B-1}}, \end{aligned} \tag{D.12}$$

and two choices of coefficients, related by analytic continuation. The first is

$$B > \frac{1}{16}, \quad \gamma > 25, \tag{D.13}$$

$$A = \frac{5}{2} - \frac{1}{2}\sqrt{(16B-1)(\gamma-25)}. \tag{D.14}$$

The second is

$$B < \frac{1}{16}, \quad \gamma < 25, \tag{D.15}$$

$$A = \frac{5}{2} + \frac{1}{2}\sqrt{(16B-1)(\gamma-25)}. \tag{D.16}$$

Note that we always have $a > 5/2$ so that $s > 0$ in (D.12).

We conclude that, even if $AdS_5 \times S^1$ is not a solution to the one-loop beta-function equations for Weyl invariance, when we consider all the α' corrections to the next order, there might be a solution. This fact points towards the possibility of having a two loop conformal invariant non-critical superstring on $AdS_5 \times S^1$. However, we have not actually proven that this choice of the coefficients solve the full supergravity equation. To show this, one would have to check that this solution satisfies also the usual gravity constraints (namely the vanishing of the stress tensor), which must be taken into account if we honestly consider the supergravity equations of motion and not just their reduction (D.11). Additionally, the coefficients A, B, and γ computed exactly using string theory may not fall withing the specified range. Moreover, the non-critical supergravity does not provide a consistent approximation to the type II non-critical superstring, as we pointed out in section 5.1. This conjectured non-critical superstring would be dual to a four-dimensional $\mathcal{N} = 2$ superconformal field theory. Indeed, the supersymmetries and the global symmetries match on the two sides.

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